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ON

LIFE CONTINGENCIES

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cap 9= tema 1 pag 191
cap 10 = tema 2 pag 21
cap 11 = tema 3 pag 2
cap 12 = tema 4 pag 24
cap 13= tema 5 pag 25

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Part II
MULTI-LIFE FUNCTIONS

Chapter 9

THE JOINT-LIFE STATUS

1. Introduction

The mathematical theory of life contingencies has been developed in Part I in terms of functions of a single life only. This chapter and the succeeding chapters in Part II are devoted to an expansion of the basic theory to cover the realm of multi-life functions which provide the means of solving problems involving two or more lives in combination.

The theory is developed from a concept of group survival. Consider a body of m lives with ages $x_1, x_2, \cdots, x_m$. Looked upon as a single collective entity, this body of lives can be said to "survive" at least as long as all of its component members survive; it becomes a matter of choice whether we say that it ceases to exist upon the first death among its members, or when the last survivor dies, or at some intermediate point. In practical applications, the component lives which are combined in an annuity or an insurance are often associated in some common undertaking, or are perhaps related by blood or marriage, and through their mutual activities may be simultaneously exposed to many of the same mortality hazards. For development of the general theory, however, the survival probabilities are assumed to be independent. *Joint-life* functions, as discussed in this chapter, are defined on the assumption that the group terminates its existence when the first death occurs. Thus the joint-life status $(x_1x_2 \cdots x_m)$ continues in existence as long as *all* the lives (x_1), (x_2), $\cdots$, (x_m) survive, and fails upon the occurrence of the *first* death.

It will be seen that this joint-life concept is precisely the type of status involved in the practical situation of a joint-life insurance payable upon the first death among the insureds, or a joint-life annuity payable as long as all the annuitants survive.

2. Joint-life probabilities

Let us consider how the probabilities of failure and survival for a joint-life status may be obtained. Although single-life mortality is measured by means of statistical studies, it is obviously im-

practical to attempt to obtain joint-life mortality rates by similar observation of groups of lives. This recourse is in fact unnecessary, since the probabilities of joint survival can be easily calculated from single-life mortality tables.

The probability that the joint-life status $(x_1x_2 \cdots x_m)$ will survive for n years is denoted by ${}_np_{x_1x_2\cdots x_m}$, and, since joint survival requires the individual survival of all the component lives, we have

$$ {}_np_{x_1x_2\cdots x_m} = {}_np_{x_1} \cdot {}_np_{x_2} \cdots {}_np_{x_m} . \tag{9.1} $$

The other basic probabilities then follow in terms of ${}_np_{x_1x_2\cdots x_m}$. The probability that the joint-life status will fail within n years is

$$ {}_nq_{x_1x_2\cdots x_m} = 1 - {}_np_{x_1x_2\cdots x_m} . \tag{9.2} $$

The probability that the status will fail in the $(n + 1)$-th year is

$$ {}_{n|}q_{x_1x_2\cdots x_m} = {}_np_{x_1x_2\cdots x_m} - {}_{n+1}p_{x_1x_2\cdots x_m} . \tag{9.3} $$

The usual annuity and insurance functions may now be defined for the joint-life status. For example, consider an immediate life annuity under which a payment of 1 is made at the end of each year as long as (x_1), (x_2), $\cdots$, (x_m) are all surviving, that is, as long as the status $(x_1x_2 \cdots x_m)$ survives. The present value of this benefit is denoted by $a_{x_1x_2\cdots x_m}$:

$$ a_{x_1x_2\cdots x_m} = \sum_{t=1}^{\infty} v^t {}_tp_{x_1x_2\cdots x_m} . $$

The joint-life insurance, with present value $A_{x_1x_2\cdots x_m}$, provides a payment of 1 at the end of the year in which the status $(x_1x_2 \cdots x_m)$ fails:

$$ A_{x_1x_2\cdots x_m} = \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q_{x_1x_2\cdots x_m} . $$

Other joint-life functions may be formulated by replacing the single-life probabilities in the corresponding single-life functions by the appropriate joint-life probabilities. For example,

$$ e_{xy} = \sum_{t=1}^{\infty} {}_tp_{xy} $$

$$ \bar{a}_{xyz} = \int_0^{\infty} v^t {}_tp_{xyz} \, dt $$

$$(IA)_{wxyz} = \sum_{t=0}^{\infty} (t+1)v^{t+1} \cdot {}_{t|}q_{wxyz} .$$

3. Joint-life mortality and commutation functions

The formula ${}_np_x = \dfrac{l_{x+n}}{l_x}$ has the formal joint-life counterpart

$${}_np_{x_1x_2\cdots x_m} = \frac{l_{x_1+n:x_2+n:\cdots:x_m+n}}{l_{x_1x_2\cdots x_m}}, \tag{9.4}$$

where $l_{x_1x_2\cdots x_m}$ remains to be defined.
Also

$${}_np_{x_1x_2\cdots x_m} = {}_np_{x_1} \cdot {}_np_{x_2} \cdots {}_np_{x_m} = \frac{l_{x_1+n}l_{x_2+n}\cdots l_{x_m+n}}{l_{x_1}l_{x_2}\cdots l_{x_m}} .$$

From these two expressions for ${}_np_{x_1x_2\cdots x_m}$, we find that

$$\frac{l_{x_1+n:x_2+n:\cdots:x_m+n}}{l_{x_1x_2\cdots x_m}} = \frac{l_{x_1+n}l_{x_2+n}\cdots l_{x_m+n}}{l_{x_1}l_{x_2}\cdots l_{x_m}},$$

and this suggests that we define the joint-life function $l_{x_1x_2\cdots x_m}$ as proportional to the product of the single-life l's; i.e.,

$$l_{x_1x_2\cdots x_m} = kl_{x_1}l_{x_2}\cdots l_{x_m} . \tag{9.5}$$

In practice, the constant of proportionality k is generally taken to be some power of 10^{-1}, so chosen as to reduce the product values to a desired order of magnitude.

Then

$$\begin{aligned} q_{x_1x_2\cdots x_m} &= 1 - p_{x_1x_2\cdots x_m} \\ &= \frac{l_{x_1x_2\cdots x_m} - l_{x_1+1:x_2+1:\cdots:x_m+1}}{l_{x_1x_2\cdots x_m}} = \frac{d_{x_1x_2\cdots x_m}}{l_{x_1x_2\cdots x_m}} \end{aligned}$$

where, by definition,

$$d_{x_1x_2\cdots x_m} = l_{x_1x_2\cdots x_m} - l_{x_1+1:x_2+1:\cdots:x_m+1} . \tag{9.6}$$

It is important to note that $d_{x_1x_2\cdots x_m}$ is *not* the same as $kd_{x_1}d_{x_2}\cdots d_{x_m}$.

Just as

$$\mu_{x+t} = -\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt} = -\frac{d \log l_{x+t}}{dt},$$

so for the joint-life status we have, by definition,

$$\mu_{x_1+t:x_2+t:\cdots:x_m+t} = -\frac{1}{l_{x_1+t:x_2+t:\cdots:x_m+t}} \frac{dl_{x_1+t:x_2+t:\cdots:x_m+t}}{dt} = -\frac{d \log l_{x_1+t:x_2+t:\cdots:x_m+t}}{dt}. \tag{9.7}$$

But this may be expressed in terms of single-life l's as

$$-\frac{d \log kl_{x_1+t} l_{x_2+t} \cdots l_{x_m+t}}{dt}$$

$$= -\frac{d \log l_{x_1+t}}{dt} - \frac{d \log l_{x_2+t}}{dt} - \cdots - \frac{d \log l_{x_m+t}}{dt}$$

$$= \mu_{x_1+t} + \mu_{x_2+t} + \cdots + \mu_{x_m+t}.$$

Thus the joint-life force of mortality is equal to the sum of the single-life forces of mortality:

$$\mu_{x_1 x_2 \cdots x_m} = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}. \tag{9.8}$$

Commutation functions for the joint-life status are defined by analogy with those for single lives; thus,

$$D_{x_1 x_2 \cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}} l_{x_1 x_2 \cdots x_m}. \tag{9.9}$$

By using the exponent $\frac{x_1 + x_2 + \cdots + x_m}{m}$, the various lives in the status are made to enter the function symmetrically, and an increase of one year in the age of each life produces an increase of unity in the exponent, so that the function behaves mathematically like its single-life counterpart.

Similarly, by definition,

$$C_{x_1 x_2 \cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}+1} d_{x_1 x_2 \cdots x_m}. \tag{9.10}$$

The functions N, S, M, and R are defined by summation in the standard way.

Since these functions bear the same relation to the joint-life mortality functions as the corresponding single-life commutation functions bear to the basic single-life mortality functions, it follows that each joint-life benefit will have a formula in commutation

symbols which exactly parallels the single-life formula for the same benefit. As illustrations, we may write

$$a_{x_1x_2\cdots x_m} = \frac{N_{x_1+1:x_2+1:\cdots:x_m+1}}{D_{x_1x_2\cdots x_m}}$$

$$A_{x_1x_2\cdots x_m} = \frac{M_{x_1x_2\cdots x_m}}{D_{x_1x_2\cdots x_m}}.$$

Many cases arise in practice in which it is not reasonable to assume that all the individual lives of a joint-life status are subject to the same single-life mortality table. Suppose, for example, that the joint two-life status (xy) is composed of the life (x) subject to mortality table T and the life (y) subject to mortality table T'. The joint-life functions of this status will reflect the dual mortality basis if the values of l_{xy} are computed as $k \cdot l_x \cdot l'_y$, using l_x from table T and l'_y from table T'.

It is important to observe that the functions $l_{x_1x_2\cdots x_m}$ and $d_{x_1x_2\cdots x_m}$ are mathematical abstractions which are even more highly idealized than their single-life counterparts l_x and d_x. Whereas the latter functions lend themselves readily to an interpretation in terms of numbers of lives surviving and dying, the functions $l_{x_1x_2\cdots x_m}$ and $d_{x_1x_2\cdots x_m}$, if they are to be given a physical meaning, must be regarded as numbers of *groups* of lives existing and failing, where existence and failure are construed in the special sense applicable to the joint-life status. Since, however, their usefulness in joint-life theory requires only that they satisfy certain mathematical definitions, any physical interpretation is irrelevant, and the student should regard them as having the nature of commutation functions, that is, purely as mathematical tools for simplifying the formulation of a particular class of problem.

4. Joint-life functions under Makeham's law

The problem of tabulating numerical values for joint-life functions presents certain obvious difficulties. A single function, even in the simplest case of two lives, requires a very extensive table if provision is made for all combinations of ages. In the case of tables following Makeham's or Gompertz's law, a remarkable simplification is possible and the volume of tabulated data can be consider-

ably reduced. We shall examine first the effect of a Makeham assumption.

Under Makeham's law, the probability of survival for a single life is given by

$$ {}_np_x = s^n g^{c^x(c^n-1)}. $$

Then for a joint-life status $(x_1x_2 \cdots x_m)$, the probability of joint survival will be

$$ {}_np_{x_1x_2\cdots x_m} = s^{mn} g^{(c^{x_1}+c^{x_2}+\cdots+c^{x_m})(c^n-1)}, \tag{9.11} $$

and, by substituting

$$ c^{x_1} + c^{x_2} + \cdots + c^{x_m} = mc^w, $$

we obtain

$$ {}_np_{x_1x_2\cdots x_m} = s^{mn} g^{mc^w(c^n-1)} = {}_np_{ww\cdots w}. $$

Thus in calculating probabilities of survival, the status $(x_1x_2 \cdots x_m)$ involving m lives of unequal age can be replaced by the simpler m-life status $(ww \cdots w)$ where all the components are of the same age. Furthermore, there is a simple relation for the determination of the "equivalent equal age" w in the condition

$$ mc^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}. \tag{9.12} $$

This condition may also be expressed in terms of the force of mortality, for, if we multiply through by B and add mA to both sides, A and B being the usual Makeham constants, we obtain

$$ m(A + Bc^w) = (A + Bc^{x_1}) + (A + Bc^{x_2}) + \cdots + (A + Bc^{x_m}), $$

whence

$$ m\mu_w = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}. \tag{9.13} $$

Thus, the equivalent equal age w may be described as the age for which the single-life force of mortality is equal to the arithmetic mean of the single-life forces of mortality at the individual ages $x_1, x_2, \cdots, x_m$.

When joint-life functions are tabulated on a Makehamized basis, the values of μ_x or of c^x are customarily shown for all ages. From these, values of w can be obtained by interpolation using formula (9.12) or (9.13).

This property of Makeham's law has the important consequence that any joint-life function, regardless of the ages involved, may be evaluated as the same function of joint lives of equal age. For, if a group of m lives of unequal ages may be replaced by m lives of equal age in calculating probabilities of survival, the same replacement may be made in any joint-life function depending on those probabilities. Thus, since ${}_np_{x_1x_2\cdots x_m} = {}_np_{ww\cdots w}$, it follows that $a_{x_1x_2\cdots x_m} = a_{ww\cdots w}$, $A_{x_1x_2\cdots x_m} = A_{ww\cdots w}$, $P_{x_1x_2\cdots x_m} = P_{ww\cdots w}$, and so on. Under these circumstances, it is only necessary to tabulate joint-life functions for equal ages, the extensive tables required for $l_{x_1x_2\cdots x_m}$, $d_{x_1x_2\cdots x_m}$, $D_{x_1x_2\cdots x_m}$, and the other functions being now replaced by *single* columns of $l_{xx\cdots x}$, $d_{xx\cdots x}$, $D_{xx\cdots x}$, and so on.

In the past many important mortality tables were constructed on a Makeham basis in order to gain this advantage. The 1941 CSO table follows Makeham's law between ages 15 and 95. Table 5 shows certain annuity values on this basis, together with values of c^x.

As an example, suppose that the value of $\ddot{a}_{30:35:45}$ is required on the 1941 CSO 2½% basis. Using Table 5, we first compute

$$c^w = \frac{c^{30} + c^{35} + c^{45}}{3} = 26.477.$$

Then, by linear interpolation in the table, we find $w = 38.363$, and hence

$$\ddot{a}_{30:35:45} = \ddot{a}_{38.363:38.363:38.363}.$$

Finally, a linear interpolation between $x = 38$ and $x = 39$ in the table of values of $\ddot{a}_{xxx}$ produces the required annuity value of 15.27735. In this result, we cannot expect all five decimal places to be correct in view of the interpolation procedure, but the loss of accuracy is slight enough to be unimportant in most applications.

There is a minor theoretical error which should be noted, although it has very little practical importance. The function which represents l_x in Makeham's law is positive for all values of x, implying an infinite limiting age in the mortality table. The 1941 CSO table used above, although Makehamized over most of its range, has l_{100} equal to 0. If we recognize this limiting age in computing the life annuity value above, we have, since ${}_tp_{45} = 0$ for $t > 54$,

TABLE 5

VALUES FROM A MAKEHAM TABLE
1941 CSO 2½%

x	a_{xx}	a_{xxx}	c^x	x
30	20.63897	18.48172	12.954	30
31	20.27405	18.11153	14.109	31
32	19.90413	17.73708	15.366	32
33	19.52954	17.35885	16.736	33
34	19.15026	16.97680	18.228	34
35	18.76695	16.59181	19.852	35
36	18.37945	16.20371	21.622	36
37	17.98838	15.81326	23.549	37
38	17.59385	15.42063	25.648	38
39	17.19596	15.02593	27.934	39
40	16.79545	14.63009	30.424	40
41	16.39232	14.23310	33.136	41
42	15.98714	13.83565	36.089	42
43	15.58011	13.43796	39.306	43
44	15.17162	13.04049	42.809	44
45	14.76233	12.64403	46.625	45
46	14.35245	12.24878	50.781	46
47	13.94244	11.85524	55.307	47
48	13.53291	11.46409	60.236	48
49	13.12403	11.07548	65.605	49
50	12.71666	10.69038	71.453	50
51	12.31105	10.30898	77.822	51
52	11.90776	9.93186	84.758	52
53	11.50720	9.55942	92.313	53
54	11.11010	9.19240	100.541	54
55	10.71676	8.83102	109.502	55
56	10.32771	8.47580	119.262	56
57	9.94354	8.12727	129.892	57
58	9.56463	7.78572	141.470	58
59	9.19149	7.45161	154.079	59
60	8.82458	7.12530	167.812	60
61	8.46435	6.80713	182.770	61
62	8.11113	6.49731	199.060	62
63	7.76562	6.19648	216.803	63
64	7.42783	5.90444	236.127	64

$$\ddot{a}_{30:35:45} = \sum_{t=0}^{54} v^t {}_tp_{30:35:45} = \sum_{t=0}^{54} v^t {}_tp_{www} = \ddot{a}_{www:\overline{55}|}$$

where $w = 38.363$. For greater accuracy then, $\ddot{a}_{30:35:45}$ should be evaluated as $\ddot{a}_{www:\overline{55}|}$ instead of as $\ddot{a}_{www}$. The difference between these two values, however, involves a small number of annuity payments discounted over a long period and contingent on the joint survival of three lives to advanced ages. It can be shown in this particular case that the error is too small to affect the fifth decimal place. Although this error can safely be ignored in practice, the student should be aware of its existence.

5. The law of uniform seniority

In the case of a joint *two*-life status, the equivalent equal age under Makeham's law may be determined in a particularly simple way. We know that the status $(x:x + n)$ can be replaced by the equal-age status $(x + t:x + t)$ provided that

$$2c^{x+t} = c^x + c^{x+n}.$$

Removal of the factor c^x in this relation yields

$$2c^t = 1 + c^n,$$

whence

$$t = \frac{\log (1 + c^n) - \log 2}{\log c}.$$

The result indicates that the seniority t of the substituted lives over the age of the younger of the original lives depends only on the difference n in the original ages. This makes it possible to construct a simple table showing the value t of this uniform seniority for each difference in ages n. When such a table is available, it is unnecessary to refer to the values of μ_x or of c^x in order to determine the age w.

The table of uniform seniority for the 1941 CSO basis is shown in Table 6. Because of the special methods used in the construction of the 1941 CSO table at the youngest and the oldest ages, the table of uniform seniority does not apply accurately if either age is under 15 or over 90.

Suppose that we require the equivalent equal age on this basis for the status (30:40). Since the difference in ages is 10, we enter Table 6 with $n = 10$ and find $t = 6.036$. Adding this value of t to the younger age 30, we find that the equivalent equal age is 36.036.

TABLE 6

UNIFORM SENIORITY

1941 CSO TABLE

Difference of Age s	Addition to Younger Age t
1	0.511
2	1.043
3	1.596
4	2.170
5	2.765
6	3.380
7	4.015
8	4.670
9	5.344
10	6.036
11	6.747
12	7.474
13	8.218
14	8.978
15	9.753
16	10.543
17	11.346
18	12.163
19	12.992
20	13.833
21	14.685
22	15.547
23	16.420
24	17.301
25	18.191

Does not apply if either age is under 15 or over 90.

The principle may be extended to the case of three joint lives, but the process is more complicated, and it is simpler to interpolate for w in a table of μ_x or c^x.

The case of four joint lives can be treated, however, by means of the table of uniform seniority for two lives. Consider first the special case of the status $(xxyy)$, involving four lives but only two distinct ages. The equivalent equal age w must satisfy the

condition

$$4c^w = 2c^x + 2c^y,$$

but this is the same as the condition

$$2c^w = c^x + c^y$$

for which (xy) is equivalent to (ww). Hence the equivalent equal age w for $(xxyy)$ is the same as for (xy).

The case of four lives of unequal age $(uvxy)$ may be reduced to the preceding case. First the equivalent equal age for each of the pairs (uv) and (xy) is found from the table of uniform seniority. Suppose that $(uv) = (rr)$ and $(xy) = (ss)$. The joint four-life status now becomes $(rrss)$, and, as above, the equivalent equal age for (rs) will be the equivalent equal age for $(rrss)$, and hence for $(uvxy)$. The ages r and s are normally fractional, so that the last step involves interpolation in the table of uniform seniority. In this case, the method is not necessarily more convenient than the process of averaging the values of μ_x or c^x.

In the case of the two-life status the table of uniform seniority makes possible a precise determination of w, since it gives values based on an exact mathematical formula. In addition, it has the practical advantage of eliminating the calculation involved in interpolating w from the table of values of c^x or μ_x.

Even when a mortality table is not Makehamized, it is sometimes possible to adopt a table of uniform seniority that will produce joint-life values for unequal ages with sufficient accuracy for many practical purposes. Although the 1958 CSO table is not a Makeham table, extensive test calculations have shown that a table of uniform seniority with $\log_{10} c = .04$ can be used to produce interpolated values for annuities, net annual premiums, and reserves which are reasonably close to the true values. The table of uniform seniority is given in Appendix I. The degree of accuracy in the resulting joint-life values depends upon the ages involved and the particular function being computed. This method should not be used with other tables without adequate testing, and if a high degree of precision is necessary, the method cannot be used.

6. Joint-life functions under Gompertz's law

With Gompertz's law, the evaluation of joint-life functions can be simplified still further. Since

$$ {}_np_{x_1x_2\cdots x_m} = g^{(c^{x_1}+c^{x_2}+\cdots+c^{x_m})(c^n-1)}, \tag{9.14} $$

by substituting $c^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}$,

we obtain

$$ {}_np_{x_1x_2\cdots x_m} = g^{c^w(c^n-1)} = {}_np_w . $$

Thus, the joint-life status $(x_1x_2 \cdots x_m)$ can be replaced by the single life (w), and joint-life functions can be evaluated directly from single-life tables, it being only necessary to determine first the age of the equivalent single life from the relation

$$ c^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m} \tag{9.15} $$

or

$$ \mu_w = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m} = \mu_{x_1x_2\cdots x_m} . \tag{9.16} $$

In this case, w is the age at which the single-life force of mortality is equal to the force of mortality for the original joint-life status. It should be noted that the substituted single life (w) is always older than any of the original lives (x_i), whereas with Makeham's law the equivalent equal age is intermediate in value to the original ages.

A law of uniform seniority holds also for Gompertz's law. The joint two-life status $(x{:}x + n)$ may be replaced by the single-life status $(x + t)$ provided that

$$ c^{x+t} = c^x + c^{x+n} $$

or

$$ c^t = 1 + c^n . $$

Hence,

$$ t = \frac{\log (1 + c^n)}{\log c} . $$

The seniority of the substituted life can be measured equally well from the older age, and, where this is done, values of $t - n$ are tabulated for each value of n.

In the Gompertz case, the same table of uniform seniority may be used with any number of lives by simply replacing pairs of joint lives by single lives until the original age status is reduced to the age of a single life. The result is independent of the order in which the lives are combined in pairs. This extension to more than two lives is not of great importance since the age of the equivalent single life is normally fractional, and it becomes necessary to in-

terpolate in the table. It is usually easier to interpolate in the table of μ_x or of c^x.

Table 7 shows annuity values at 3% for a Gompertz table together with the table of uniform seniority. Suppose that the value of $a_{50:55}$ is required. We enter the table of uniform seniority with a difference in ages of 5 and obtain a seniority of 6.858, and, noting that the table measures seniority from the older life, we have (50:55) = (61.858). Then, by interpolation in the column of annuity values,

$$a_{50:55} = a_{61.858} = 11.70476.$$

If the value of $a_{50:55:65}$ is also required, we interpolate in the table of uniform seniority for a difference in ages of $65 - 61.858 = 3.142$ to find (61.858:65) = (72.645). Hence,

$$a_{50:55:65} = a_{72.645} = 7.88446.$$

7. Summary of notation

1. The special symbol $(x_1x_2 \cdots x_m)$ denotes a joint-life status in which the individual lives are aged $x_1, x_2, \cdots, x_m$. When, for the sake of distinctness, it is desired to separate the ages in such a status, a colon is placed between them; e.g., (20:25), $(x + t:y + t)$.

2. When a suffixed subscript consists of two or more ages without any distinguishing mark, a joint-life status is intended. Thus: l_{xy}, $a_{20:30:40}$.

3. Brackets are used to denote select mortality; e.g., $A_{[40][50]}$, $l_{[x]+t:[y]+t}$.

4. When a joint-life status is involved and the absence of any distinguishing mark would lead to ambiguity, the symbol — is placed above the lives included—as in $A^{\overline{1}}_{\overline{xy}:\overline{n}|}$.

5. *Commutation functions*

$$D_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}} l_{x_1x_2\cdots x_m}$$

$$\bar{D}_{x_1x_2\cdots x_m} = \int_0^1 D_{x_1+t:x_2+t:\cdots:x_m+t}\, dt$$

$$C_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}+1} d_{x_1x_2\cdots x_m}$$

$$\bar{C}_{x_1x_2\cdots x_m} = \int_0^1 D_{x_1+t:x_2+t:\cdots:x_m+t}\mu_{x_1+t:x_2+t:\cdots:x_m+t}\, dt$$

The functions N, $\bar{N}$, S, $\bar{S}$, M, $\bar{M}$, R, $\bar{R}$ are obtained by summation in the usual way.

TABLE 7

Uniform Seniority in a Gompertz Table

Annuity Values at 3%		Table of Uniform Seniority	
x	a_x	Difference of Ages	Addition to Older Age
40	19.52384	0	9.122
41	19.19752	1	8.632
42	18.86680	2	8.160
43	18.53179	3	7.707
44	18.19268	4	7.273
45	17.84964	5	6.858
46	17.50283	6	6.461
47	17.15249	7	6.082
48	16.79884	8	5.721
49	16.44204	9	5.377
50	16.08243	10	5.050
51	15.72020	11	4.739
52	15.35566	12	4.445
53	14.98909	13	4.166
54	14.62080	14	3.902
55	14.25107	15	3.653
56	13.88028	16	3.417
57	13.50872	17	3.195
58	13.13677	18	2.986
59	12.76477	19	2.789
60	12.39309	20	2.604
61	12.02211	21	2.430
62	11.65224	22	2.266
63	11.28381	23	2.113
64	10.91726	24	1.970
65	10.55296	25	1.835
66	10.19131	26	1.709
67	9.83271	27	1.591
68	9.47754	28	1.481
69	9.12620	29	1.378
70	8.77904	30	1.282
71	8.43646	31	1.192
72	8.09883	32	1.109
73	7.76647	33	1.031
74	7.43975	34	.958

TABLE 7—*Continued*

Annuity Values at 3%		Table of Uniform Seniority	
x	a_x	Difference of Ages	Addition to Older Age
75	7.11896	35	.890
76	6.80441	36	.827
77	6.49642	37	.768
78	6.19521	38	.714
79	5.90105	39	.663
80	5.61416	40	.615

References

4. The values of c^x shown in Table 5 are taken from Heckman (1946). This paper also gives joint-life mortality and commutation functions on the 1941 CSO basis with interest at 2½% and 3%.

When Makeham's law holds, joint-life annuities can be computed from a single-life table with a change in the rate of interest. The method is described by Jones (1930) and forms the basis of Exercises 11 and 12.

The theoretical error in replacing a_{xy} by a_{ww} is pointed out by Gershenson (1957).

5. The data of Table 6 are taken from Thompson (1941). The extension of this method to three joint lives requires two tables of uniform seniority. It is illustrated by Spurgeon (1932), p. 259.

For the 1958 CSO table, the complete table of uniform seniority and the tests of accuracy for various functions may be seen in the Report of the General Committee on Publication of Monetary Tables, TSA 11, pp. 1049 ff.

6. The data of Table 7 are based on the 1937 Standard Annuity Table, described by Kineke (1938).

A general treatment of laws of mortality which satisfy a uniform seniority principle is given by Greville (1956).

EXERCISES

1, 2. Joint-life probabilities

1. Express each of the following in terms of the single-life probabilities ${}_np_x$ and ${}_np_y$:

(a) the probability that (xy) will survive n years;
(b) the probability that exactly one of the lives (x) and (y) will survive n years;
(c) the probability that at least one of the lives (x) and (y) will survive n years;
(d) the probability that (xy) will fail within n years;
(e) the probability that at least one of the lives will die within n years;
(f) the probability that both lives will die within n years.

2. Express in terms of ${}_n|q_x$, ${}_n|q_y$, and ${}_n|q_z$:
(a) the probability that all three lives (x), (y), and (z) will die in the $(n + 1)$-th year;
(b) the probability that none of the lives will die in the $(n + 1)$-th year;
(c) the probability that at least one of the lives will die in the $(n + 1)$-th year.

3. Show that the probability that (x) survives n years and (y) survives $n - 1$ years may be expressed either as

$$\frac{{}_np_{x:y-1}}{p_{y-1}} \quad \text{or as} \quad p_x \cdot {}_{n-1}p_{x+1:y} .$$

4. Show that the probability that two lives (30) and (40) will die at the same age last birthday can be expressed as

$${}_{10}p_{30}(1 + e_{40:40}) - 2\,{}_{11}p_{30}(1 + e_{40:41}) + p_{40} \cdot {}_{11}p_{30}(1 + e_{41:41}).$$

3. Joint-life mortality and commutation functions

5. Since $l_{xy} = kl_xl_y$, a student assumes that the following relations are correct:
(a) $d_{xy} = kd_xd_y$
(b) $\mu_{xy} = k\mu_x\mu_y$
(c) $p_{xy} = p_xp_y$
(d) $q_{xy} = q_xq_y$

Which of these are incorrect? Give the correct expressions.

6. If a special mortality table is derived from a standard table by the relationship $\mu_x' = k\mu_x$ (k integral), show that ${}_tp_x' = {}_tp_{xx\cdots(k)}$.

7. Show that
(a) $D_{xy} = k(1 + i)^{\frac{x+y}{k}} D_x \cdot D_y$
(b) $C_{xy} = vD_{xy} - D_{x+1:y+1}$
(c) $D_{xy} = \sqrt{D_{xx} \cdot D_{yy}}$

8. Find the derivatives with respect to x of:
(a) $\log {}_tp_{xx}$ (b) $\mathring{e}_{xx}$.

4. Joint-life functions under Makeham's law

9. Use the fact that the 1941 CSO table is Makehamized to evaluate the following functions on the 1941 CSO 2½% basis:

(a) $a_{40:\overline{50}}$ (Ans. 13.33810)
(b) $A_{40:\overline{50}}$ (Ans. .65029)
(c) $P_{40:\overline{50}}$ (Ans. .04535)
(d) $\bar{a}_{30:\overline{40:50}}$ (Ans. 13.56522)

Explain how you would proceed if the 1941 CSO table were not Makehamized.

10. Two single-life tables, one for male and one for female lives, both follow Makeham's law and have a common value of the constant c. From these two tables, a joint two-life table is constructed applying to pairs of lives one of which is male and the other female.

Prove that by this joint-life table for a male life (x) and a female life (y)

$$a_{xy} = a_{ww}$$

where
$$c^w = \frac{c^x + kc^y}{1 + k},$$

k being equal to
$$\frac{\log g'}{\log g} = \frac{B'}{B},$$

the primed symbols applying to the original female table and unprimed symbols to the original male table.

11. Assuming Makeham's law, show that

$${}_np_{xy} = {}_nE_w,$$

where $c^w = c^x + c^y$ and ${}_nE_w$ is calculated at rate of interest $i = \frac{1}{s} - 1$, s being the usual Makeham constant.

12. Prove that if a mortality table is constructed according to Makeham's law the value of an annuity on m joint lives aged $x, y, z, \cdots$ calculated at a rate of interest i is equal to the value of a single-life annuity at age w calculated at a rate of interest i' where

$$i' = \frac{1 + i}{s^{m-1}} - 1.$$

Show that the age w may be determined from the relation

$$\mu_{w+h} = \frac{\mu_x + \mu_y + \mu_z + \cdots}{m}$$

where
$$h = -\frac{\log m}{\log c}.$$

5. The law of uniform seniority

13. (a) Use the table of uniform seniority to evaluate $a_{40:\overline{50}}$ on the 1941 CSO 2½% basis. Compare your answer with that obtained in exer-

cise 9(a) and explain why there is not an exact agreement in the two results. (Ans. 13.33769)

(b) Use the tables in Appendix I to compute an approximate value for $a_{40:50}$ on the 1958 CSO 3% basis. (Ans. 14.08285)

14. Show that the law of uniform seniority holds for two lives when the law of mortality is

$$\mu_x = a + bx.$$

15. What is the formula for calculating the table of uniform seniority for the joint-life table of Exercise 10?

6. Joint-life functions under Gompertz's law

16. (a) Calculate $a_{60:70}$ from Table 7. (Ans. 7.10323)

(b) For the calculation of $a_{50:55:65}$ shown in the text, verify that the same result is obtained if the lives (55) and (65) are combined first.

17. With Gompertz's law, if (z) is the single life which is equivalent to the joint status (xy), show that there is a theoretical inaccuracy in writing $a_z = a_{xy}$ for a table which has a finite limiting age ω. What is the correct relationship? Why can the inaccuracy generally be ignored in practice?

18. (a) A mortality table for male lives follows Makeham's law, and another table for female lives follows Gompertz's law with the constants B and c the same as for the male table. Show that in the evaluation of a joint-life annuity on m male lives and n female lives, all of unequal ages, m male lives of equal age can be substituted for the $m + n$ lives.

(b) Show how the table of uniform seniority may be constructed for the case where $m = n = 1$.

Miscellaneous problems

19. If the Fackler valuation functions $u_{xx} = \dfrac{D_{xx}}{D_{x+1:x+1}}$ and $k_{xx} = \dfrac{C_{xx}}{D_{x+1:x+1}}$ are computed at a certain age y on a certain mortality basis with $i = .03$, it is found that $u_{yy} = 4\ k_{yy}$. Find p_y. (Ans. .86)

20. Compute $\mathring{e}_{25:25}$ if $\mu_x = \dfrac{1}{100 - x}$, $0 < x < 100$. (Ans. 25)

21. You are given the following probabilities:

(i) The probability that three persons aged 20, 30 and 40 will live 10 years is .758.

(ii) The probability that a person aged 45 will die within 5 years, while another person aged 40 is alive at the end of 5 years, is .063.

(iii) The probability that four persons aged 20, 25, 30, and 35 will live for 5 years, while a fifth person aged 40 will not be alive at the end of 5 years, is .045.

Find the probability that a person aged 20 will be alive at the end of 25 years. (Ans. .812)

22. A mortality table is graduated according to Makeham's law: $\mu_x = A + Bc^x$. A new mortality table is constructed such that

$$\mu'_x = A' + B'c^x,$$

where

$$A' = -\frac{\delta}{2} \quad \text{and} \quad B' = Bc^{\frac{1}{2}}.$$

Express a_{xy}, based on the new mortality table, as a single-life function derived from the Gompertz table which is a specialization of the original Makeham table.

CHAPTER 10

LAST-SURVIVOR AND GENERAL MULTI-LIFE STATUSES

1. The last-survivor status

In the last chapter we noted that a body of m lives, considered as a single collective entity, can be said to fail upon the first death among its component lives, or upon the death of its last surviving member, or arbitrarily upon some intermediate death. The concept of group failure upon the first death leads to the joint-life functions; the major alternative, that of defining the failure of the group as occurring when its last survivor dies, leads to the *last-survivor* status and the various functions of that status. The expressions "longest-life" and "joint-and-survivor" are also used in insurance practice as synonymous with "last-survivor."

The symbol $(\overline{x_1 x_2 \cdots x_m})$ denotes a group of m lives aged x_1, x_2, $\cdots$, x_m, comprising a last-survivor status. The group continues to exist as long as *at least one* of (x_1), (x_2), $\cdots$, (x_m) is alive, and fails upon the occurrence of the *last* death.

The symbol ${}_np_{\overline{x_1 x_2 \cdots x_m}}$ denotes the probability that the last-survivor status $(\overline{x_1 x_2 \cdots x_m})$ will survive for n years. This function may also be described as the probability that at least one of the lives (x_1), (x_2), $\cdots$, (x_m) will survive for n years, or as the probability that the last survivor will be alive at the end of n years. There is, of course, no way of knowing at the outset which one of the lives will be the last survivor. It should be understood that the language, "the last survivor will be alive," does not imply that none of the other lives will also be surviving; it is merely an alternative way of saying that at least one of the lives will be in existence.

It is evident that ${}_np_{\overline{x_1 x_2 \cdots x_m}}$ may be expressed as the complement of the probability that all the lives will die within n years. Thus,

$$
\begin{aligned}
{}_np_{\overline{x_1 x_2 \cdots x_m}} &= 1 - (1 - {}_np_{x_1})(1 - {}_np_{x_2}) \cdots (1 - {}_np_{x_m}) \\
&= ({}_np_{x_1} + {}_np_{x_2} + \cdots + {}_np_{x_m}) \\
&\quad - ({}_np_{x_1 x_2} + {}_np_{x_1 x_3} + \cdots + {}_np_{x_{m-1} x_m})
\end{aligned}
$$

$$+ \left({}_np_{x_1x_2x_3} + {}_np_{x_1x_2x_4} + \cdots + {}_np_{x_{m-2}x_{m-1}x_m}\right)$$
$$- \left({}_np_{x_1x_2x_3x_4} + \cdots + {}_np_{x_{m-3}x_{m-2}x_{m-1}x_m}\right)$$
$$+ \cdots$$
$$+ (-1)^{m+1}\,{}_np_{x_1x_2\ldots x_m}.$$

This expression may be written more concisely as follows:

$${}_np_{\overline{x_1x_2\cdots x_m}} = \sum {}_np_{x_1} - \sum {}_np_{x_1x_2} + \sum {}_np_{x_1x_2x_3} - \sum {}_np_{x_1x_2x_3x_4} + \cdots + (-1)^{m+1}\,{}_np_{x_1x_2\ldots x_m} \tag{10.1}$$

where the summations are understood to cover all possible combinations of the lives taken one at a time, two at a time, three at a time, and so on. It is thus evident that the probabilities of survival for the last-survivor status may be obtained in terms of single-life and joint-life probabilities.

The probabilities of failure for the status $(\overline{x_1x_2\cdots x_m})$ may now be defined in terms of ${}_np_{\overline{x_1x_2\cdots x_m}}$. The probability that the status will fail within n years is

$${}_nq_{\overline{x_1x_2\cdots x_m}} = 1 - {}_np_{\overline{x_1x_2\cdots x_m}}. \tag{10.2}$$

The probability that $(\overline{x_1x_2\cdots x_m})$ will fail in the $(n+1)$-th year is given by

$${}_{n|}q_{\overline{x_1x_2\cdots x_m}} = {}_np_{\overline{x_1x_2\cdots x_m}} - {}_{n+1}p_{\overline{x_1x_2\cdots x_m}}. \tag{10.3}$$

The most common last-survivor functions are those which involve only two lives or three lives. In these cases, assuming that the lives are (x), (y), and (z), we have from (10.1)

$${}_np_{\overline{xy}} = \sum {}_np_x - {}_np_{xy} = {}_np_x + {}_np_y - {}_np_{xy} \tag{10.4}$$

$${}_np_{\overline{xyz}} = \sum {}_np_x - \sum {}_np_{xy} + {}_np_{xyz}$$
$$= {}_np_x + {}_np_y + {}_np_z - {}_np_{xy} - {}_np_{xz} - {}_np_{yz} + {}_np_{xyz}. \tag{10.5}$$

Other functions can be expressed in a similar way; for example,

$${}_nq_{\overline{xy}} = 1 - {}_np_{\overline{xy}} = 1 - ({}_np_x + {}_np_y - {}_np_{xy})$$
$$= {}_nq_x + {}_nq_y - {}_nq_{xy}$$
$$= \sum {}_nq_x - {}_nq_{xy}$$

$$
\begin{aligned}
{}_{n|}q_{\overline{xyz}} &= {}_{n}p_{\overline{xyz}} - {}_{n+1}p_{\overline{xyz}} \\
&= ({}_{n}p_x + {}_{n}p_y + {}_{n}p_z - {}_{n}p_{xy} - {}_{n}p_{xz} - {}_{n}p_{yz} + {}_{n}p_{xyz}) - \\
&\quad ({}_{n+1}p_x + {}_{n+1}p_y + {}_{n+1}p_z - {}_{n+1}p_{xy} - {}_{n+1}p_{xz} - {}_{n+1}p_{yz} + {}_{n+1}p_{xyz}) \\
&= {}_{n|}q_x + {}_{n|}q_y + {}_{n|}q_z - {}_{n|}q_{xy} - {}_{n|}q_{xz} - {}_{n|}q_{yz} + {}_{n|}q_{xyz} \\
&= \sum {}_{n|}q_x - \sum {}_{n|}q_{xy} + {}_{n|}q_{xyz} \\
a_{\overline{xyz}} &= \sum_{t=1}^{\infty} v^t {}_{t}p_{\overline{xyz}} \\
&= \sum_{t=1}^{\infty} v^t ({}_{t}p_x + {}_{t}p_y + {}_{t}p_z - {}_{t}p_{xy} - {}_{t}p_{xz} - {}_{t}p_{yz} + {}_{t}p_{xyz}) \\
&= a_x + a_y + a_z - a_{xy} - a_{xz} - a_{yz} + a_{xyz} \\
&= \sum a_x - \sum a_{xy} + a_{xyz}
\end{aligned}
$$

In each of the special cases considered above for two lives and for three lives, it will be observed that the form of the result is the same as the form of (10.4) or (10.5). It will be apparent upon reflection that this is not merely coincidental, but that the similarity follows necessarily from the manner in which each of the functions depends upon the probabilities of survival. The functions, ${}_{n}q$, ${}_{n|}q$, and a, can all be expressed[1] as linear combinations of the values of ${}_{t}p$. If we let $F({}_{t}p)$ denote such a linear function of ${}_{t}p$, it will follow from the property of linearity that

$$
\begin{aligned}
F({}_{t}p_{\overline{x_1x_2\cdots x_m}}) &= F(\sum {}_{t}p_{x_1} - \sum {}_{t}p_{x_1x_2} + \cdots + (-1)^{m+1} {}_{t}p_{x_1x_2\cdots x_m}) \\
&= F(\sum {}_{t}p_{x_1}) - F(\sum {}_{t}p_{x_1x_2}) + \cdots + (-1)^{m+1} F({}_{t}p_{x_1x_2\cdots x_m}) \\
&= \sum F({}_{t}p_{x_1}) - \sum F({}_{t}p_{x_1x_2}) + \cdots + (-1)^{m+1} F({}_{t}p_{x_1x_2\cdots x_m}).
\end{aligned}
$$

Hence, any last-survivor function, $F_{\overline{x_1x_2\cdots x_m}}$, which can be written as a linear combination of values of ${}_{t}p$, may be expressed in terms of single-life functions F_{x_1} and joint-life functions $F_{x_1x_2}$, $F_{x_1x_2x_3}$, $\cdots$, $F_{x_1x_2\cdots x_m}$, in accordance with the relation

$$
F_{\overline{x_1x_2\cdots x_m}} = \qquad (10.6)
$$

$$
\sum F_{x_1} - \sum F_{x_1x_2} + \sum F_{x_1x_2x_3} - \cdots + (-1)^{m+1} F_{x_1x_2\cdots x_m}.
$$

[1] The function ${}_{n}q$ becomes linear in ${}_{t}p$ when written as ${}_{0}p - {}_{n}p$, where ${}_{0}p = 1$.

The net single premiums for the common annuity and insurance benefits can normally be expressed as linear combinations of the probabilities of survival, and relation (10.6) therefore holds in these cases. Examples of functions which are *not* linear in ${}_tp$ are net annual premiums and reserves. For example,

$$P_{\overline{xy}} = \frac{A_{\overline{xy}}}{\ddot{a}_{\overline{xy}}} = \frac{A_x + A_y - A_{xy}}{\ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}},$$

which is clearly different from $P_x + P_y - P_{xy}$. For the reserve value, the expression to be used depends upon which lives are still in existence at the time of valuation. Thus, in the case of the insurance $A_{\overline{xy}}$, the t-th year terminal reserve will be

$$A_{\overline{x+t:y+t}} - P_{\overline{xy}} \cdot \ddot{a}_{\overline{x+t:y+t}}$$

if (x) and (y) are both alive,

$$A_{x+t} - P_{\overline{xy}} \cdot \ddot{a}_{x+t}$$

if only (x) is alive,

or

$$A_{y+t} - P_{\overline{xy}} \cdot \ddot{a}_{y+t}$$

if only (y) is alive.

2. Compound statuses

Both the joint-life status and the last-survivor status have been defined only for groups of single lives. It is possible, however, and frequently useful, to relate these concepts to groups of individual statuses which are not necessarily single lives. For example, the annuity $a_{\overline{wx}:\overline{yz}}$ is payable during the *joint* existence of the *last survivor* of (w) and (x) and the *last survivor* of (y) and (z); the compound status $(\overline{wx}:\overline{yz})$ is a *joint* status of which the individual components are themselves *last-survivor* statuses. As another example, the insurance benefit represented by $A_{\overline{wx}:\overline{yz}}$ is payable when the second failure of (wx) and $(\overline{yz})$ has occurred, so that $(\overline{wx}:\overline{\overline{yz}})$ is a compound last-survivor status of which the components are a joint-life status and a last-survivor status.

Functions of this kind can always be expressed in terms of single-life and simple joint-life functions. The process will be illustrated for the two cases cited above.

$$a_{\overline{wx}:\overline{yz}} = \sum_{t=1}^{\infty} v^t {}_tp_{\overline{wx}:\overline{yz}}$$

$$= \sum_{t=1}^{\infty} v^t {}_tp_{\overline{wx}} \cdot {}_tp_{\overline{yz}} \text{ (since both } (\overline{wx}) \text{ and } (\overline{yz}) \text{ must survive)}$$

$$= \sum_{t=1}^{\infty} v^t ({}_tp_w + {}_tp_x - {}_tp_{wx})({}_tp_y + {}_tp_z - {}_tp_{yz})$$

$$= \sum_{t=1}^{\infty} v^t ({}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} - {}_tp_{wyz} - {}_tp_{xyz} - {}_tp_{wxy} - {}_tp_{wxz} + {}_tp_{wxyz})$$

$$= a_{wy} + a_{wz} + a_{xy} + a_{xz} - a_{wyz} - a_{xyz} - a_{wxy} - a_{wxz} + a_{wxyz}$$

$$A_{\overline{wx}:\overline{yz}} = \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q_{\overline{wx}:\overline{yz}} = \sum_{t=0}^{\infty} v^{t+1} ({}_tp_{\overline{wx}:\overline{yz}} - {}_{t+1}p_{\overline{wx}:\overline{yz}})$$

$$= \sum_{t=0}^{\infty} v^{t+1} \{[1 - (1 - {}_tp_{wx})(1 - {}_tp_{\overline{yz}})] - [1 - (1 - {}_{t+1}p_{wx})(1 - {}_{t+1}p_{\overline{yz}})]\}$$

(since ${}_tp_{\overline{wx}:\overline{yz}}$ is the complement of the probability that both (wx) and $(\overline{yz})$ will fail in t years)

$$= \sum_{t=0}^{\infty} v^{t+1} \{[1 - (1 - {}_tp_{wx})(1 - {}_tp_y - {}_tp_z + {}_tp_{yz})] - [1 - (1 - {}_{t+1}p_{wx})(1 - {}_{t+1}p_y - {}_{t+1}p_z + {}_{t+1}p_{yz})]\}$$

$$= \sum_{t=0}^{\infty} v^{t+1} [({}_tp_y + {}_tp_z - {}_tp_{yz} + {}_tp_{wx} - {}_tp_{wxy} - {}_tp_{wxz} + {}_tp_{wxyz}) - ({}_{t+1}p_y + {}_{t+1}p_z - {}_{t+1}p_{yz} + {}_{t+1}p_{wx} - {}_{t+1}p_{wxy} - {}_{t+1}p_{wxz} + {}_{t+1}p_{wxyz})]$$

$$= \sum_{t=0}^{\infty} v^{t+1} ({}_{t|}q_y + {}_{t|}q_z - {}_{t|}q_{yz} + {}_{t|}q_{wx} - {}_{t|}q_{wxy} - {}_{t|}q_{wxz} + {}_{t|}q_{wxyz})$$

$$= A_y + A_z - A_{yz} + A_{wx} - A_{wxy} - A_{wxz} + A_{wxyz}.$$

While it is of fundamental importance for the understanding of the subject that the student be able to resolve compound expres-

sions of this type into their single- and joint-life elements, it will be noted that in practice the expansions required can become tedious. Certain general relationships between last-survivor and joint-life statuses make it possible to reduce very markedly the steps needed for these expansions.

Consider first the function $a_{\overline{uv}}$, where (u) and (v) represent any type of status thus far defined—single-life, joint-life, last-survivor, or term-certain. Whatever the individual statuses (u) and (v) may be, we know that the last-survivor annuity $a_{\overline{uv}}$ will provide annual payments of 1 for a period determined by the *longer* surviving status. On the other hand, the joint-life annuity a_{uv} provides payments only during the existence of the *shorter* surviving status. Hence, $a_{\overline{uv}}$ provides an annuity payable during the continuance of one of the statuses (u) and (v), and a_{uv} provides an annuity on the other. It follows that

$$a_{\overline{uv}} + a_{uv} = a_u + a_v ,$$

whence

$$a_{\overline{uv}} = a_u + a_v - a_{uv} . \tag{10.7a}$$

It will be observed that the two-life case of relation (10.6) applied to annuities is merely a special case of the above result. Similar reasoning produces an analogous relation for the present values of insurances:

$$A_{\overline{uv}} = A_u + A_v - A_{uv} . \tag{10.7b}$$

Next consider the function $a_{u:\overline{vw}}$, where again the statuses (u), (v), (w) are not necessarily single lives. This function provides annuity payments for as long as (u) survives jointly with the *longer* surviving of the two statuses (v) and (w). The function a_{uvw}, on the other hand, provides payments for as long as (u) survives jointly with the *shorter* surviving of the two statuses (v) and (w). Hence, $a_{u:\overline{vw}}$ is payable during the continuance of one of the statuses (uv) and (uw), and a_{uvw} provides an annuity on the other. It follows that

$$a_{u:\overline{vw}} + a_{uvw} = a_{uv} + a_{uw} ,$$

whence

$$a_{u:\overline{vw}} = a_{uv} + a_{uw} - a_{uvw} . \tag{10.8a}$$

The corresponding result for insurances is

$$A_{u:\overline{vw}} = A_{uv} + A_{uw} - A_{uvw} . \qquad (10.8b)$$

The two examples discussed earlier may now be used to illustrate how these general relationships reduce the number of steps needed to obtain the required expansion. Thus, for the function $a_{\overline{wx}:\overline{yz}}$, we use (10.8a):

$$a_{\overline{wx}:\overline{yz}} = a_{\overline{wx}:y} + a_{\overline{wx}:z} - a_{\overline{wx}:yz} .$$

Applying (10.8a) a second time,

$$a_{\overline{wx}:\overline{yz}} =$$

$$(a_{wy} + a_{xy} - a_{wxy}) + (a_{wz} + a_{xz} - a_{wxz}) - (a_{wyz} + a_{xyz} - a_{wxyz}),$$

agreeing with the previous result.

Similarly, $A_{\overline{wx:yz}}$ is first expanded by (10.7b) to

$$A_{wx} + A_{\overline{yz}} - A_{wx:\overline{yz}} ,$$

and then expanding $A_{\overline{yz}}$ and applying (10.8b) to the term $A_{wx:\overline{yz}}$, we find

$$A_{\overline{wx:yz}} = A_{wx} + (A_y + A_z - A_{yz}) - (A_{wxy} + A_{wxz} - A_{wxyz}),$$

as before.

It should be noted that the general reasoning by which formulas (10.7) and (10.8) have been derived is equally valid when the compound status involves a term-certain. Consider an annuity payable until the death of (x) or until (y) attains age $y + n$, whichever period is longer. The two statuses are (x) and $(y:\overline{n}|)$, and the annuity is payable during the continuance of the last-surviving status. We thus have, from (10.7a),

$$a_{\overline{(x)(y:\overline{n}|)}} = a_x + a_{y:\overline{n}|} - a_{xy:\overline{n}|} .$$

If the annuity is payable until (x) attains age $x + n$ or for m years certain, whichever is longer, the statuses are $(x:\overline{n}|)$ and $(\overline{m}|)$, and we have $a_{\overline{(x:\overline{n}|)(\overline{m}|)}} = a_{x:\overline{n}|} + a_{\overline{m}|} - a_{x:\overline{n}|:\overline{m}|}$. For $m < n$, the status $(\overline{m}|)$ fails before the status $(\overline{n}|)$, and hence the joint status $(x:\overline{n}|:\overline{m}|)$ is the same as $(x:\overline{m}|)$. The function $a_{x:\overline{n}|:\overline{m}|}$ can be written in this case as $a_{x:\overline{m}|}$. We thus have

$$a_{\overline{(x:\overline{n}|)(\overline{m}|)}} = a_{x:\overline{n}|} + a_{\overline{m}|} - a_{x:\overline{m}|} , \qquad m < n.$$

An annuity payable until either (x) attains age $x + n$ or (y) attains age $y + m$ $(m < n)$, whichever period is longer, has the present value

$$a_{\overline{(x:\overline{n}|)(y:\overline{m}|)}} = a_{x:\overline{n}|} + a_{y:\overline{m}|} - a_{\overline{xy:\overline{n}|:\overline{m}|}}$$
$$= a_{x:\overline{n}|} + a_{y:\overline{m}|} - a_{xy:\overline{m}|}, \qquad m < n.$$

3. The general multi-life status

The joint-life status and the last-survivor status may be regarded as special cases of a general multi-life status. Let the m lives, (x_1), (x_2), $\cdots$, (x_m), comprise a status which is defined as surviving as long as at least r of the individual lives survive and which therefore fails upon the occurrence of the $(m - r + 1)$-th death. This general status is denoted by the symbol $(\overline{x_1x_2 \cdots x_m}^{r})$. When r equals m, this reduces to the joint-life status; when r equals 1, a last-survivor status is defined.

The symbol ${}_np_{\overline{x_1x_2\cdots x_m}}^{r}$ denotes the probability that the status $(\overline{x_1x_2 \cdots x_m}^{r})$ will survive for n years, that is, that at least r out of the original m lives will still be in existence n years hence. In order to evaluate this function, it is first necessary to find an expression for the probability that *exactly* r lives will be surviving at the end of n years, denoted by ${}_np_{\overline{x_1x_2\cdots x_m}}^{[r]}$.

If the m lives are all of the same age x, the probability that exactly r *designated* lives will be surviving at the end of n years, the other $m - r$ being dead, is clearly

$$({}_np_x)^r(1 - {}_np_x)^{m-r}.$$

Now there are $\binom{m}{r}$ ways of selecting a group of r lives out of m, and the probability that *any* group of exactly r lives will be surviving, and the others dead, is

$${}_np_{\overline{xx\cdots x}}^{[r]} = \binom{m}{r}({}_np_x)^r(1 - {}_np_x)^{m-r}. \tag{10.9}$$

Note that this expression is equivalent to the coefficient of t^r in the binomial expansion of $({}_np_x \cdot t + {}_nq_x)^m$.

Suppose now that the m lives are not all of the same age. The probability that exactly r specified lives, (x_1), (x_2), $\cdots$, (x_r), will be surviving at the end of n years and the others dead is

$$ {}_np_{x_1} \cdot {}_np_{x_2} \cdots {}_np_{x_r}(1 - {}_np_{x_{r+1}})(1 - {}_np_{x_{r+2}}) \cdots (1 - {}_np_{x_m}). $$

The probability ${}_np_{\overline{x_1x_2\cdots x_m}}^{[r]}$ is the sum of all such expressions containing exactly r factors ${}_np_{x_i}$ and exactly $m - r$ factors $1 - {}_np_{x_j}$. To find this sum, we write the following product:

$$ ({}_np_{x_1} \cdot t + {}_nq_{x_1})({}_np_{x_2} \cdot t + {}_nq_{x_2}) \cdots ({}_np_{x_m} \cdot t + {}_nq_{x_m}). $$

In the expansion of this product, the coefficient of t^r is the sum of all the terms which can be formed by multiplying r factors of the form ${}_np_{x_i}$ and $m - r$ factors of the form ${}_nq_{x_j}$, and is thus equal to the required sum for ${}_np_{\overline{x_1x_2\cdots x_m}}^{[r]}$. We rewrite the product as

$$ [1 + (t - 1)\,{}_np_{x_1}][1 + (t - 1)\,{}_np_{x_2}] \cdots [1 + (t - 1)\,{}_np_{x_m}] $$
$$ = 1 + (t - 1)\,Z_1 + (t - 1)^2 Z_2 + \cdots + (t - 1)^m Z_m, $$

where Z_s denotes the summation $\sum {}_np_{x_1x_2\ldots x_s}$ taken over all the possible combinations of s lives out of m. Now t^r appears only in the terms involving Z_s where $s \geqslant r$. The coefficient of t^r in the term involving Z_s, that is, the term $(t - 1)^s Z_s$ $(s \geqslant r)$, is

$$ (-1)^{s-r}\binom{s}{s-r}Z_s. $$

Hence the total coefficient of t^r is

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{[r]} = \sum_{s=r}^{m} (-1)^{s-r}\binom{s}{s-r}Z_s = Z_r - \binom{r+1}{1}Z_{r+1} $$
$$ + \binom{r+2}{2}Z_{r+2} - \cdots + (-1)^{m-r}\binom{m}{m-r}Z_m. \quad (10.10) $$

The presence of the binomial coefficients in this formula suggests the following formal device. If we regard the indices of Z as exponents and assume that $Z^t = 0$ for $t > m$, the formula may be written in the following symbolic form:

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{[r]} = \frac{Z^r}{(1 + Z)^{r+1}}. \quad (10.11) $$

As an example of the use of formula (10.11), we obtain the probability that exactly two of the four lives, (w), (x), (y), (z), will survive for n years. Since $m = 4$ and $r = 2$, we first write

$$ \frac{Z^2}{(1 + Z)^3} = Z^2 - 3Z^3 + 6Z^4. $$

Then,

$$ {}_np_{\overline{wxyz}}^{[2]} = {}_np_{wx} + {}_np_{wy} + {}_np_{wz} + {}_np_{xy} + {}_np_{xz} + {}_np_{yz} $$
$$ - 3\,({}_np_{wxy} + {}_np_{wxz} + {}_np_{wyz} + {}_np_{xyz}) + 6\,{}_np_{wxyz}\,. $$

A partial check can be obtained by letting ${}_np = 1$ for each single life. We then find

$$ {}_np_{\overline{wxyz}}^{[2]} = 6 - 3\cdot(4) + 6 = 0. $$

This is clearly correct since if each life is certain to survive, the probability that exactly two will survive is 0.

We now return to the probability ${}_np_{\overline{x_1x_2\cdots x_m}}^{r}$ that *at least* r lives will survive for n years. We shall show that this probability can be expressed as follows:

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{r} = \sum_{s=r}^{m} (-1)^{s-r}\binom{s-1}{s-r}Z_s = Z_r - \binom{r}{1}Z_{r+1} $$
$$ + \binom{r+1}{2}Z_{r+2} - \cdots + (-1)^{m-r}\binom{m-1}{m-r}Z_m\,. \quad (10.12) $$

The proof is by induction on r. When $r = 1$, the formula is

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{1} = Z_1 - Z_2 + Z_3 - \cdots + (-1)^{m-1}Z_m\,. $$

Since this is equivalent to (10.1), we see that the formula is valid in this case. We now show that if the formula holds for $r = k$, it also holds for $r = k + 1$.

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{k+1} = {}_np_{\overline{x_1x_2\cdots x_m}}^{k} - {}_np_{\overline{x_1x_2\cdots x_m}}^{[k]} $$
$$ = \sum_{s=k}^{m} (-1)^{s-k}\binom{s-1}{s-k}Z_s - \sum_{s=k}^{m} (-1)^{s-k}\binom{s}{s-k}Z_s $$
$$ = \sum_{s=k}^{m} (-1)^{s-k}\left[\binom{s-1}{s-k} - \binom{s}{s-k}\right]Z_s $$
$$ = \sum_{s=k+1}^{m} (-1)^{s-k-1}\binom{s-1}{s-k-1}Z_s\,, $$

where the expression $\binom{s-1}{s-k} - \binom{s}{s-k}$ has been replaced by $-\binom{s-1}{s-k-1}$ for $s \geqslant k + 1$ and by 0 for $s = k$. This relation is familiar from Pascal's triangle of the binomial coefficients; it can be easily derived as an algebraic identity. Since the result is the same as (10.12) for $r = k + 1$, the induction is complete and we conclude that (10.12) holds for all positive integers r. The formula may be written in symbolic form as

$$ {}_{n}p_{\overline{x_1x_2\cdots x_m}}^{\ r} = \frac{Z^r}{(1+Z)^r}. \tag{10.13} $$

As an example of the use of formula (10.13), we obtain the probability that at least two of the three lives, (x), (y), (z), will be alive at the end of n years. Since $m = 3$ and $r = 2$, we first write

$$ \frac{Z^2}{(1+Z)^2} = Z^2 - 2Z^3. $$

Then,

$$ {}_{n}p_{\overline{xyz}}^{\ 2} = {}_{n}p_{xy} + {}_{n}p_{xz} + {}_{n}p_{yz} - 2\,{}_{n}p_{xyz}. $$

As a check in this case, if we let ${}_{n}p = 1$ for each single life, we have

$$ {}_{n}p_{\overline{xyz}}^{\ 2} = 1 + 1 + 1 - 2 = 1. $$

If each life is certain to survive, the probability that at least two will survive is 1.

The expression $\dfrac{Z^r}{(1+Z)^r}$ of formula (10.13) may also be used to expand any linear function of the probability ${}_{n}p_{\overline{x_1x_2\cdots x_m}}^{\ r}$ in terms of the corresponding joint-life functions. For example,

$$ a_{\overline{xyz}}^{\ 2} = a_{xy} + a_{xz} + a_{yz} - 2a_{xyz} $$

$$ A_{\overline{wxyz}}^{\ 2} = A_{wxy} + A_{wxz} + A_{wyz} + A_{xyz} - 3A_{wxyz}. $$

When the insurance symbol is used in this way, it must be kept in mind that $A_{\overline{x_1x_2\cdots x_m}}^{\ r}$ represents an insurance payable upon the failure of the status $\overline{(x_1x_2\cdots x_m)}^{r}$. The insurance, therefore, is payable not at the r-th death but at the $(m-r+1)$-th death.

Similarly, the expression $\dfrac{Z^r}{(1+Z)^{r+1}}$ of formula (10.11) may be used to expand linear functions of ${}_{n}p_{\overline{x_1x_2\cdots x_m}}^{\ [r]}$. As an example, the present value of an annuity of 1 payable at the end of each year if exactly one of the three lives, (x), (y), and (z), is then surviving is given by

$$ a_{\overline{xyz}}^{\ [1]} = a_x + a_y + a_z - 2(a_{xy} + a_{xz} + a_{yz}) + 3a_{xyz}. $$

It should be noted that this benefit is a form of deferred annuity, no payments being made until two deaths have occurred. The expression can be checked by noting the annuity payments that are provided while the different numbers of lives are surviving. When all three are alive, each of the annuity functions contributes a payment of 1, and the total payment is $1 + 1 + 1 - 2 - 2 - 2 + 3 = 0$. When exactly two are alive, there is a payment for *two* of the single-life functions and for *one* of the two-life functions, and the total payment is $1 + 1 - 2 = 0$. When exactly one is alive, there is a payment of 1, as required, from one of the single-life functions. This is also a useful device for checking the corresponding probability expansion.

4. Summary of notation

If a horizontal bar appears above a suffixed subscript, then *survivors* of the lives, and not *joint* lives, are intended. The number of survivors is denoted by a symbol over the right end of the bar. If that symbol, say r, is enclosed in brackets, the meaning is *exactly* r survivors; without brackets, the meaning is *at least* r survivors. If no symbol appears over the bar, then unity is assumed and the meaning is *at least one* survivor.

References

3. The formula for ${}_np_{\overline{x_1x_2\cdots x_m}}^{\overline{[r]}}$ was derived by Waring in 1792. In its general form, Waring's Theorem gives the probability that exactly r out of m events will occur, and thus has wide applications. The theorem is discussed in standard texts on probability, e.g., Bizley (1957). White and Greville (1959) have discussed computational devices which simplify the application of the theorem. A related method of computing the values of the individual shares in a last-survivor annuity has been given by Rasor and Myers (1952).

The proof by induction for formula (10.12) was suggested by Harry Gershenson, F.S.A.

EXERCISES

1. The last-survivor status

1. Show that $q_{\overline{xy}} = q_x \cdot q_y$, and give an alternative expression in terms of q_x, q_y, and q_{xy}.

2. Express each of the following in terms of single- and joint-life functions:

(a) $d_{\overline{wxyz}}$

(b) $P^{\frac{1}{}}_{\overline{xy}:\overline{n}|}$

3. Find the probability that at least one of the two lives (x) and (y) will die in the $(n+1)$-th year. Is this the same as ${}_{n|}q_{\overline{xy}}$? Explain.

4. Describe the benefit whose net single premium is ${}_{n|}a_{\overline{xy}}$. Show that ${}_{n|}a_{\overline{xy}}$ is *not* equivalent to $v^n{}_np_{xy}a_{\overline{x+n:y+n}}$ nor to $v^n{}_np_{\overline{xy}}a_{\overline{x+n:y+n}}$. Give a correct expression for ${}_{n|}a_{\overline{xy}}$.

5. Given a mortality table following Makeham's law and two ages x and y for which (ww) is the equivalent equal-age status.

(a) Show that ${}_tp_w$ is the geometric mean of ${}_tp_x$ and ${}_tp_y$.

(b) If $x \neq y$, show that ${}_tp_x + {}_tp_y > 2{}_tp_w$, and hence that $a_{\overline{xy}} > a_{\overline{ww}}$.

6. Given $a_{xy} = a_{ww}$ on a Makehamized mortality table, and $y < x$. Rank the following in order of magnitude: a_y, a_w, a_{xx}, a_{ww}, $a_{\overline{xy}}$, $a_{\overline{yy}}$.

2. Compound statuses

7. Describe the benefit whose net single premium is denoted by $A_{\overline{x:\overline{n}|}}$. Show that $A_{\overline{x:\overline{n}|}} = A_x - A_{x:\overline{n}|} + v^n$.

8. Show that

$$a_{\overline{wx}:\overline{yz}} = a_{\overline{wx}} + a_{\overline{yz}} - a_{\overline{wxyz}},$$

and explain by general reasoning.

9. Show that

$$a_{\overline{xy:\overline{n}|}} = a_{\overline{n}|} + {}_{n|}a_{xy}.$$

Describe this benefit.

10. Express $a_{abc:\overline{xyz}}$ in terms of joint-life annuities.

11. Express in annuity symbols the value of an annuity of 1 payable at the end of each year until the survivor of (25) or (30) reaches age 50, or until his death if both lives die before attaining that age.

12. Express in annuity symbols the value of a deferred annuity of 1 payable as long as either (25) or (30) is living after age 50.

13. Give an expression for evaluating $a_{\overline{x:y:\overline{n}|}}$ in terms of annuities-certain and single- and joint-life annuities.

3. The general multi-life status

14. Show that the symbolic formula

$${}_np_{\overline{xy\ldots z}}^{\frac{r}{}} = \frac{Z^r}{(1+Z)^r}$$

may be obtained formally by an algebraic summation over s of

$$ {}_{n}p_{\overline{xy\ldots z}}^{[s]} = \frac{Z^s}{(1+Z)^{s+1}} . $$

15. Express in terms of single- and joint-life probabilities:

(a) ${}_{n}p_{\overline{xyz}}^{\ 2}$

(b) ${}_{n}p_{\overline{xyz}}^{[1]}$

16. Describe in words the probability indicated by the following expression:

$$ {}_{t}p_{wx} + {}_{t}p_{wy} + {}_{t}p_{wz} + {}_{t}p_{xy} + {}_{t}p_{xz} + {}_{t}p_{yz} $$
$$ - 3({}_{t}p_{wxy} + {}_{t}p_{wxz} + {}_{t}p_{wyz} + {}_{t}p_{xyz}) + 6\,{}_{t}p_{wxyz} . $$

17. Give an expression in terms of joint-life annuity values for the net single premium for an annuity of 1 payable at the end of each year if exactly 3 of the 4 lives (w), (x), (y), (z) are then alive.

18. Find the value of $a_{\overline{xxxxxx}}^{[4]}$, given that

$$ a_{xxxx} = 11.900,\ a_{xxxxx} = 10.750 \text{ and } a_{xxxxxx} = 9.675. \qquad \text{(Ans. 1.125)} $$

19. Prove mathematically and by general reasoning:

$$ a_{\overline{xyz}} - a_{\overline{xyz}}^{[1]} = a_{\overline{xyz}}^{\ 2} . $$

20. Express in terms of annuity values and the rate of discount the net single premium for an insurance payable at the end of the year in which the third death occurs among the four lives (w), (x), (y), (z).

21. Obtain an expression for evaluating $\ddot{a}_{w:\overline{xyz}}^{\ 2}$, an annuity-due payable for as long as (w) survives jointly with at least two of the lives, (x), (y), and (z).

Miscellaneous problems

22. A partnership is formed by five men, all of whom are aged x exactly. Find:

(a) The probability that not more than three of the original partners will be alive at the end of n years.

(b) The probability that not more than one of the original partners will die in the n-th year.

23. An extract from a table of joint-life annuities valued at 3½% reads as follows:

Age of Lives	Net Single Premium Joint-Life Immediate Annuity
26: 20: 28	14.40
29: 26: 20	14.30
28: 29: 26	13.80
20: 28: 29	14.00
29: 26: 28: 20	12.50

Find the net single premium for an insurance of \$1000 payable at the end of the year of death of the second life to fail out of four lives aged 20, 26, 28, and 29. (Ans. \$324)

24. An annuity of 1 is payable to (x) and (y) on a last-survivor basis commencing at the end of $m + k$ years, provided that both lives survive m years. Show that its present value is

$$ {}_mp_y \cdot {}_{k+m-1|}a_x + {}_mp_x \cdot {}_{k+m-1|}a_y - {}_{k+m-1|}a_{xy} . $$

25. An annuity of 1 is payable yearly to (x) as long as he lives jointly with (y) and for n years after the death of (y) except that in no event will payments be made after m years from the present time, $m > n$. Show that its present value is

$$ a_{x:\overline{n}|} + \frac{D_{x+n}}{D_x} a_{x+n:y:\overline{m-n}|} . $$

26. An insurance of 1 is to be payable in the event that (x) dies before age $x + n$ and (y) dies before age $y + m$, where $m < n$, the face amount becoming payable at the end of the year in which the second death occurs. Show that the present value may be expressed as

$$ A^{\frac{1}{}}_{\overline{xy}:\overline{m}|} + v^m {}_mp_x(1 - {}_mp_y)A^{\frac{1}{}}_{\overline{x+m}:\overline{n-m}|} , $$

and that the net annual premium would be determined by dividing by

$$ \ddot{a}_{\overline{xy}:\overline{m}|} + v^m {}_mp_x(1 - {}_mp_y)\ddot{a}_{x+m:\overline{n-m}|} . $$

27. Last-survivor annuities often provide for a reduction in the amount of the annuity payment after the first death. An n-year temporary annuity-due payable to $(\overline{xy})$ provides annual payments of 1 while both lives survive, reducing to ½ on the death of (x) or to ⅓ on the death of (y). Express the present value in terms of annuity values.

28. An annual annuity is to be payable so long as any of the lives (w), (x), (y), and (z) survive. At each death the rate of annual payment is reduced 50% from the rate in effect prior to such death. Find a simplified expression involving only single- and joint-life annuities for the cost of the contract per unit of initial annual payment.

29. Obtain an expression in annuity symbols for the net single premium for a continuous annuity of 1 per annum payable while at least one of the two lives (40) and (50) is living and is over age 60 but not while either of them is living under age 55.

Chapter 11

CONTINGENT FUNCTIONS

1. Introduction

In the various types of multi-life status discussed in the preceding chapters, the component lives are not distinguished according to the order in which they die. The status (xyz), for example, terminates upon the first death, whether it is the death of (x), of (y), or of (z); and the status $(\overline{xyz})$ terminates upon the last death regardless of the particular order in which the individual deaths occur.

A different group of functions is required in situations where the order of deaths is significant, as, for example, in the case of an insurance payable upon the death of (x) provided that he dies before (y). Functions of this type, depending upon a specified order of deaths, are known as *contingent functions.*

The present chapter deals only with the simplest class of contingent function—that in which the order of failure is prescribed with reference to only one of the individual components involved. A subsequent chapter will describe functions in which the order of failure is prescribed for more than one of the components—the *compound* contingent functions.

2. Contingent probabilities

When the lives of a group are all of the same age and subject to the same mortality, each has the same chance of dying first, or second, and contingent probabilities are easily evaluated by reference to the possible orders of death. Thus, in the case of two lives each aged x, the probability that a specified one of them will die first is simply equal to $\frac{1}{2}$. Similarly, the probability that the specified life will die first and within n years is

$$\tfrac{1}{2} \cdot {}_nq_{xx},$$

${}_nq_{xx}$ representing the chance that the joint lifetime ends within n years, and $\frac{1}{2}$ being the chance that it is terminated by the failure of the specified life.

In general, the lives involved are not of equal age. Consider

first the case of two lives aged x and y, and let q^1_{xy} denote the probability that (x) will die before (y) and within one year—in other words, that the joint lifetime will fail within one year because of the death of (x). This probability may be expressed in definite integral form as

$$q^1_{xy} = \int_0^1 {}_tp_{xy}\mu_{x+t}\,dt = \frac{1}{l_{xy}}\int_0^1 l_{x+t:y+t}\mu_{x+t}\,dt, \tag{11.1}$$

since the differential ${}_tp_{xy}\mu_{x+t}\,dt$ represents the probability that (x) will die at the moment of attaining age $x + t$, (y) being still alive.

The other basic contingent probabilities for two lives may be written as follows:

$${}_nq^1_{xy} = \int_0^n {}_tp_{xy}\mu_{x+t}\,dt \tag{11.2}$$

$${}_\infty q^1_{xy} = \int_0^\infty {}_tp_{xy}\mu_{x+t}\,dt \tag{11.3}$$

$${}_{n|}q^1_{xy} = \int_n^{n+1} {}_tp_{xy}\mu_{x+t}\,dt. \tag{11.4}$$

Note the symbol used in (11.3) for the probability that (x) will die before (y) with no limit on the time of death.

If the specified life (x) is to die *second* within n years, the probability is

$$\begin{aligned} {}_nq^2_{xy} &= \int_0^n {}_tp_x(1 - {}_tp_y)\mu_{x+t}\,dt \\ &= \int_0^n {}_tp_x\mu_{x+t}\,dt - \int_0^n {}_tp_{xy}\mu_{x+t}\,dt \\ &= {}_nq_x - {}_nq^1_{xy}. \end{aligned} \tag{11.5}$$

The result may be obtained by general reasoning, since if (x) dies within n years he will die either before or after (y); i.e., ${}_nq_x = {}_nq^1_{xy} + {}_nq^2_{xy}$.

The probability that (x) will die second and during the $(n + 1)$-th year may similarly be seen to be

$${}_{n|}q^2_{xy} = {}_{n|}q_x - {}_{n|}q^1_{xy}. \tag{11.6}$$

Note that (y) need not die in the $(n+1)$-th year, but may die at any time preceding the death of (x).

The probabilities involving three lives, contingent upon the death of a specified life (x), are as follows:

$$ {}_nq^{1}_{xyz} = \int_0^n {}_tp_{xyz}\mu_{x+t}\,dt \tag{11.7} $$

$$ \begin{aligned} {}_nq^{2}_{xyz} &= \int_0^n {}_tp_x \cdot {}_tp^{\overline{[1]}}_{yz}\mu_{x+t}\,dt \\ &= \int_0^n {}_tp_x({}_tp_y + {}_tp_z - 2_tp_{yz})\mu_{x+t}\,dt \\ &= {}_nq^{1}_{xy} + {}_nq^{1}_{xz} - 2_nq^{1}_{xyz} \end{aligned} \tag{11.8} $$

$$ \begin{aligned} {}_nq^{3}_{xyz} &= \int_0^n (1 - {}_tp_y)(1 - {}_tp_z){}_tp_x\mu_{x+t}\,dt \\ &= {}_nq_x - {}_nq^{1}_{xy} - {}_nq^{1}_{xz} + {}_nq^{1}_{xyz}. \end{aligned} \tag{11.9} $$

The probability ${}_nq^{\overline{1}}_{\overline{xy}:z}$, that either (x) or (y) will be the first to die within n years, is an instance of a function in which the specified status is not a single life:

$$ {}_nq^{\overline{1}}_{\overline{xy}:z} = {}_nq^{1}_{xyz} + {}_nq^{1}_{xyz}\,. \tag{11.10} $$

The probability that (x) will die within n years and before the survivor of (y) and (z) is

$$ \begin{aligned} {}_nq^{1}_{x:\overline{yz}} &= \int_0^n {}_tp_{x:\overline{yz}}\mu_{x+t}\,dt \\ &= \int_0^n {}_tp_x({}_tp_y + {}_tp_z - {}_tp_{yz})\mu_{x+t}\,dt \\ &= {}_nq^{1}_{xy} + {}_nq^{1}_{xz} - {}_nq^{1}_{xyz}. \end{aligned} \tag{11.11} $$

This function may also be described as the probability that (x) will die either first or second and within n years, and an alternative formulation is

$$ {}_nq^{1:2}_{x\ yz} = {}_nq^{1}_{xyz} + {}_nq^{2}_{xyz}. \tag{11.12} $$

In view of (11.8), this latter expression may easily be seen to be equivalent to that obtained above. Another expression for this

probability is

$$ {}_nq_z - {}_nq^{\overline{2}}_{xyz} . $$

Consider the function ${}_nq^{\overline{2}}_{\overline{xy}:z}$, representing the probability that the survivor of (x) and (y) will die within n years, predeceased by (z). In formulating this probability as a definite integral, it is necessary to express the differential probability that the survivor of (x) and (y) will die at moment of time t. This expression is

$$ [{}_tp_x(1 - {}_tp_y)\mu_{x+t} + {}_tp_y(1 - {}_tp_x)\mu_{y+t}]\,dt $$

or

$$ ({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t})\,dt. $$

Hence

$$ {}_nq^{\overline{2}}_{\overline{xy}:z} = \int_0^n (1 - {}_tp_z)({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t})\,dt $$

$$ = ({}_nq_x + {}_nq_y - {}_nq_{xy}) - ({}_nq^{1}_{xz} + {}_nq^{1}_{yz} - {}_nq^{\overline{1}}_{\overline{xy}:z}). \quad (11.13) $$

3. Evaluation of contingent probabilities

In the elementary case where the lives are of equal age, contingent probabilities can be evaluated directly from general probability principles. It is sometimes helpful to write the integral expression and carry out the integration as a check on the direct reasoning.

For three lives all aged x, we know that the probability that a specified life will die first within n years is

$$ {}_nq^{1}_{xxx} = \tfrac{1}{3} \cdot {}_nq_{xxx} = \tfrac{1}{3}\,[1 - ({}_np_x)^3]. $$

Using the integral form, we have

$$ {}_nq^{1}_{xxx} = \int_0^n {}_tp_{xxx}\mu_{x+t}\,dt $$

$$ = \int_0^n ({}_tp_x)^2\,{}_tp_x\mu_{x+t}\,dt $$

$$ = -\int_{t=0}^{t=n} ({}_tp_x)^2 d({}_tp_x) $$

$$ = [-\tfrac{1}{3}\,({}_tp_x)^3]_0^n = \tfrac{1}{3}\,[1 - ({}_np_x)^3]. $$

It is important to distinguish this probability from two others that are similar.

(1) If the failing life is not specified, the probability that the first death occurs within n years is simply the probability that the joint-life status fails within n years:

$$\begin{aligned} {}_nq_{xxx} &= \int_0^n {}_tp_{xxx}\mu_{x+t:x+t:x+t}\,dt \\ &= \int_0^n {}_tp_{xxx}\cdot 3\mu_{x+t}\,dt \\ &= 3\,{}_nq^{\overline{1}}_{xxx} = 1 - ({}_np_x)^3. \end{aligned}$$

(2) The probability that a specified life dies within n years and the other two survive the n-year period is

$$\int_0^n {}_tp_x\mu_{x+t}\cdot {}_np_{xx}\,dt = {}_nq_x\cdot {}_np_{xx} = (1 - {}_np_x)({}_np_x)^2.$$

If the failing life is not specified, the probability that exactly one will die within n years is $3(1 - {}_np_x)\ ({}_np_x)^2$.

Suppose that the three lives aged x are A, B, and C. What is the probability that either A or B will be the first to die within n years? The joint lifetime of A and B must fail before C, and hence the solution is given by ${}_nq^{\overline{1}}_{\overline{xx}:x}$. Using (11.10), we find ${}_nq^{\overline{1}}_{\overline{xx}:x} = 2\,{}_nq^{\overline{1}}_{xxx} = \frac{2}{3}[1 - ({}_np_x)^3]$. The solution by integration is

$$\begin{aligned} {}_nq^{\overline{1}}_{\overline{xx}:x} &= \int_0^n {}_tp_{xx}\mu_{x+t:x+t}\cdot {}_tp_x\,dt \\ &= \int_0^n {}_tp_{xxx}\cdot 2\,\mu_{x+t}\,dt \\ &= \tfrac{2}{3}[1 - ({}_np_x)^3]. \end{aligned}$$

When these functions involve lives of unequal age, the integrations cannot be directly carried out and special methods must be used. Standard methods of numerical integration are available which can be programmed for a digital computer if a large volume of calculations must be made.

Another method of evaluation is based upon the assumption of a uniform distribution of deaths. In the formula

$$q^{\overline{1}}_{xy} = \int_0^1 {}_tp_{xy}\mu_{x+t}\,dt,$$

we substitute from formula (1.24b)

$$tp_x\mu_{x+t} \doteq q_x ,$$

and obtain

$$q^1_{xy} \doteq q_x \int_0^1 {}_tp_y \, dt \doteq \frac{d_x l_{y+\frac{1}{2}}}{l_x l_y} \tag{11.14}$$

where $l_{y+\frac{1}{2}}$ is taken as $\frac{1}{2}(l_y + l_{y+1})$. Then

$${}_nq^1_{xy} \doteq \frac{1}{l_x l_y} \sum_{t=0}^{n-1} d_{x+t} \cdot l_{y+t+\frac{1}{2}} . \tag{11.15}$$

A similar approximation can be used when more than two lives are involved:

$${}_nq^1_{xyz} \doteq \frac{1}{l_x l_y l_z} \sum_{t=0}^{n-1} d_{x+t} l_{y+t+\frac{1}{2}} l_{z+t+\frac{1}{2}} . \tag{11.16}$$

When either Gompertz's or Makeham's law applies, special formulas are available which make possible an exact evaluation. For instance, with Gompertz's law,

$$\begin{aligned}
{}_nq^1_{x_1x_2\cdots x_m} &= \int_0^n {}_tp_{x_1x_2\cdots x_m} Bc^{x_1+t} \, dt \\
&= \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}} \int_0^n {}_tp_{x_1x_2\cdots x_m} Bc^t (c^{x_1} + c^{x_2} + \cdots + c^{x_m}) \, dt \\
&= \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}} \cdot {}_nq_{x_1x_2\cdots x_m} \\
&= \frac{c^{x_1}}{c^w} \cdot {}_nq_w = \frac{\mu_{x_1}}{\mu_w} \cdot {}_nq_w ,
\end{aligned} \tag{11.17}$$

where w is the age of the equivalent single life. A special case of the above result is

$${}_\infty q^1_{x_1x_2\cdots x_m} = \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}} . \tag{11.18}$$

We may then write

$${}_nq^1_{x_1x_2\ldots x_m} = {}_\infty q^1_{x_1x_2\cdots x_m} \cdot {}_nq_{x_1x_2\ldots x_m} . \tag{11.19}$$

Thus, with Gompertz's law, the probability that (x_1) will die first within n years is equal to the probability that (x_1) will be the first to die multiplied by the probability that the joint lives will fail within n years.

A precise formula may also be obtained when Makeham's law applies. With this law, we have

$$
\begin{aligned}
{}_n{}^1q_{x_1x_2\cdots x_m} &= \int_0^n {}_tp_{x_1x_2\cdots x_m}(A + Bc^{x_1+t})\,dt \\
&= A\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt \\
&\quad + \frac{c^{x_1}}{c^{x_1}+c^{x_2}+\cdots+c^{x_m}}\int_0^n {}_tp_{x_1x_2\cdots x_m}Bc^t(c^{x_1}+c^{x_2}+\cdots+c^{x_m})\,dt \\
&= A\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt + \frac{c^{x_1}}{c^{x_1}+c^{x_2}+\cdots+c^{x_m}} \\
&\qquad \cdot \int_0^n {}_tp_{x_1x_2\cdots x_m}(\mu_{x_1+t}+\mu_{x_2+t}+\cdots+\mu_{x_m+t}-mA)\,dt \\
&= A\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt \\
&\qquad + \frac{c^{x_1}}{c^{x_1}+c^{x_2}+\cdots+c^{x_m}}\left({}_nq_{x_1x_2\cdots x_m} - mA\cdot\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt\right) \\
&= A\cdot\bar{a}_{x_1x_2\cdots x_m:\overline{n}|} + \frac{c^{x_1}}{c^{x_1}+c^{x_2}+\cdots+c^{x_m}} \\
&\qquad \cdot({}_nq_{x_1x_2\cdots x_m} - mA\cdot\bar{a}_{x_1x_2\cdots x_m:\overline{n}|}) \\
&= A\bar{a}_{ww\cdots w:\overline{n}|} + \frac{c^{x_1}}{mc^w}({}_nq_{ww\cdots w} - mA\cdot\bar{a}_{ww\cdots w:\overline{n}|})
\end{aligned}
$$

where $mc^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}$.

Replacing A by $-\log s$ and rearranging terms,

$$
\begin{aligned}
&{}_n{}^1q_{x_1x_2\cdots x_m} \\
&= \frac{c^{x_1}}{c^w}\left(\frac{1}{m}\cdot{}_nq_{ww\cdots w} + \log s\cdot\bar{a}_{ww\cdots w:\overline{n}|}\right) - \log s\cdot\bar{a}_{ww\cdots w:\overline{n}|}. \qquad (11.20)
\end{aligned}
$$

Then as a special case

$$ {}_{\infty}q^{1}_{x_1x_2\cdots x_m} = \frac{c^{x_1}}{c^{\omega}}\left(\frac{1}{m} + \log s\cdot\ell_{\omega\omega\cdots\omega}\right) - \log s\cdot\ell_{\omega\omega\cdots\omega}. \quad (11.21) $$

No special techniques are required for the evaluation of probabilities contingent upon a second or subsequent death, since these can always be expressed in terms of probabilities contingent upon the first death.

4. Contingent insurance functions

Let A^{1}_{xy} denote the net single premium for a contingent insurance of 1 payable at the end of the year of death of (x) provided that (x) dies before (y). Then

$$ A^{1}_{xy} = \sum_{t=0}^{\omega} v^{t+1}\ {}_{t|}q^{1}_{xy}. \quad (11.22) $$

When the value of A^{1}_{xy} is known, the complementary function $A^{1}_{\overline{xy}}$ is easily obtained from the relation

$$ A^{1}_{xy} + A^{1}_{\overline{xy}} = A_{xy}. $$

For the contingent term insurance, we have

$$ A^{1}_{xy:\overline{n}|} = \sum_{t=0}^{n-1} v^{t+1}\ {}_{t|}q^{1}_{xy}. \quad (11.23) $$

The payment is made at the end of the year of death of (x), provided that (x) dies before (y) and within n years—or, in other words, if the status (x) is the first to fail of the three statuses, (x), (y), and $(\overline{n}|)$. It may be noted that the case where (x) is the first to fail of the *two* statuses, (x) and $(\overline{n}|)$, is merely the familiar single-life function $A^{1}_{x:\overline{n}|}$.

When the insurance is payable at the moment of death, instead of at the end of the year of death, the formulas appear in definite integral form.

$$ \bar{A}^{1}_{xy} = \int_0^{\infty} v^t\ {}_tp_{xy}\mu_{x+t}\,dt \quad (11.24) $$

$$ \bar{A}^{1}_{xy:\overline{n}|} = \int_0^{n} v^t\ {}_tp_{xy}\mu_{x+t}\,dt \quad (11.25) $$

A few examples involving three lives follow:

$$\bar{A}^{1}_{xyz} = \int_0^\infty v^t {}_tp_{xyz}\mu_{x+t}\,dt \tag{11.26}$$

$$\bar{A}^{\frac{1}{xy}:z} = \int_0^\infty v^t {}_tp_{xyz}\mu_{x+t:y+t}\,dt = \int_0^\infty v^t {}_tp_{xyz}(\mu_{x+t} + \mu_{y+t})\,dt$$

$$= \bar{A}^{1}_{xyz} + \bar{A}^{1}_{xyz} \tag{11.27}$$

$$\bar{A}^{\frac{1}{xy}:z} = \int_0^\infty v^t {}_tp_z({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t})\,dt$$

$$= \bar{A}^{1}_{xz} + \bar{A}^{1}_{yz} - \bar{A}^{\frac{1}{xy}:z}$$

$$= \bar{A}^{1}_{xz} + \bar{A}^{1}_{yz} - \bar{A}^{1}_{xyz} - \bar{A}^{1}_{xyz}. \tag{11.28}$$

Note that the contingent functions in this last result can all be made to depend upon (z) dying first:

$$\bar{A}^{\frac{1}{xy}:z} = (\bar{A}_{xz} - \bar{A}^{1}_{xz}) + (\bar{A}_{yz} - \bar{A}^{1}_{yz}) - (\bar{A}_{xyz} - \bar{A}^{1}_{xyz}). \tag{11.29}$$

Contingent insurances can also be defined when the specified status is not the *first* to fail. For example, the symbol A^{2}_{xy} denotes the net single premium for an insurance of 1 payable at the end of the year of death of (x) provided that (y) has died before (x):

$$A^{2}_{xy} = \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q^{2}_{xy} = \sum_{t=0}^{\infty} v^{t+1}({}_{t|}q_x - {}_{t|}q^{1}_{xy}) = A_x - A^{1}_{xy}. \tag{11.30}$$

The result is evident from general reasoning, since $A^{1}_{xy} + A^{2}_{xy} = A_x$. Similarly,

$$\bar{A}^{2}_{xy} = \int_0^\infty v^t(1 - {}_tp_y){}_tp_x\mu_{x+t}\,dt = \bar{A}_x - \bar{A}^{1}_{xy}. \tag{11.31}$$

The definite integral which appears in (11.31) can be transformed in an interesting way. We first note that the expression $1 - {}_tp_y$ can itself be written as the definite integral ${}_tq_y = \int_0^t {}_sp_y\mu_{y+s}\,ds$, and this leads to a double integral for $\bar{A}^{2}_{xy}$:

$$\bar{A}^{2}_{xy} = \int_0^\infty \int_0^t v^t {}_tp_x\mu_{x+t}\,{}_sp_y\mu_{y+s}\,ds\,dt.$$

By inverting the order of integration, we obtain the following equivalent expression:

$$\bar{A}^2_{xy} = \int_0^\infty \int_s^\infty v^t \, {}_tp_x \mu_{x+t} \, {}_sp_y \mu_{y+s} \, dt \, ds.$$

Performing the integration with respect to t, we have

$$\bar{A}^2_{xy} = \int_0^\infty {}_sp_y \mu_{y+s} \, {}_{s|}\bar{A}_x \, ds$$

$$= \int_0^\infty v^s \, {}_sp_{xy} \mu_{y+s} \, \bar{A}_{x+s} \, ds. \tag{11.32}$$

In this form, the function appears as a varying insurance on the life of (y), contingent on the survival of (x). The amount of the varying insurance at any time is the net single premium for insurance of 1 on (x) at his then attained age.

It is not necessary in general to derive separate formulas for contingent functions of both the A and $\bar{A}$ type as we have done above for A^2_{xy} and $\bar{A}^2_{xy}$ in (11.30) and (11.31). The relations among the net single premiums are the same whether the insurance is payable at the end of the year of death or at the moment of death.

It will be found that the net single premium for any contingent insurance, when written as a definite integral, can readily be transformed to an expression which is solely in terms of single- and joint-life functions and contingent functions dependent on the *first* death. This is illustrated in the examples that follow:

$$\bar{A}^2_{xyz} = \int_0^\infty v^t \, {}_tp_x \cdot {}_tp^{[1]}_{yz} \mu_{x+t} \, dt \tag{11.33}$$

$$= \bar{A}^1_{xy} + \bar{A}^1_{xz} - 2\bar{A}^1_{xyz}$$

$$\bar{A}^{\,2}_{\overline{xy}:z} = \int_0^\infty v^t (1 - {}_tp_z) \, {}_tp_{xy} \mu_{x+t:y+t} \, dt \tag{11.34}$$

$$= \bar{A}_{xy} - \bar{A}^{\,1}_{\overline{xy}:z} = \bar{A}_{xy} - \bar{A}^1_{xyz} - \bar{A}^{\,1}_{xy\overline{z}}$$

$$\bar{A}^{\,2}_{xy\overline{z}} = \int_0^\infty v^t (1 - {}_tp_y)(1 - {}_tp_z) \, {}_tp_x \mu_{x+t} \, dt \tag{11.35}$$

$$= \bar{A}_x - \bar{A}^1_{xy} - \bar{A}^1_{xz} + \bar{A}^1_{xyz}.$$

A special notation is used when these insurances are on a temporary basis, in that the term period is indicated by a prefixed subscript. For example, an insurance on the life of (x) payable if he dies after (y) and within n years is represented by ${}_n\bar{A}^2_{xy}$.

It should be noted that the symbol $A^2_{xy:\overline{n}|}$ denotes an insurance payable upon the death of (x) provided that (x) is the second of the three statuses, (x), (y), $(\overline{n}|)$, to fail, and this benefit is different from the term insurance described above. It is thus important that symbols like ${}_nA^2_{xy}$ and $A^2_{xy:\overline{n}|}$ be correctly distinguished.

5. Evaluation of contingent insurances

When it is necessary to evaluate a large number of contingent insurance functions on the same mortality and interest basis, the construction of special commutation columns will simplify the calculations. These are usually based on the approximation afforded by the assumption of a uniform distribution of deaths,

$$ {}_t|q^1_{xy} \doteq \frac{d_{x+t} \cdot l_{y+t+\frac{1}{2}}}{l_x \cdot l_y}, $$

so that

$$ A^1_{xy} = \sum_{t=0}^{\infty} v^{t+1} {}_t|q^1_{xy} \doteq \sum_{t=0}^{\infty} \frac{v^{\frac{x+y}{2}+t+1} d_{x+t} \cdot l_{y+t+\frac{1}{2}}}{v^{\frac{x+y}{2}} l_x \cdot l_y} $$

$$ = \frac{\sum_{t=0}^{\infty} C^1_{x+t:y+t}}{D_{xy}} = \frac{M^1_{xy}}{D_{xy}}, $$

where $$ C^1_{xy} = v^{\frac{x+y}{2}+1} k d_x l_{y+\frac{1}{2}} \qquad \text{with } k = \frac{l_{xy}}{l_x \cdot l_y} \tag{11.36} $$

and $$ M^1_{xy} = \sum_{t=0}^{\infty} C^1_{x+t:y+t}. \tag{11.37} $$

For the contingent term insurance, we have

$$ A^1_{xy:\overline{n}|} = \frac{M^1_{xy} - M^1_{x+n:y+n}}{D_{xy}}. $$

The exact definition of C^1_{xy} is

$$ C^1_{xy} = v^{\frac{x+y}{2}+1} \int_0^1 l_{x+t:y+t} \mu_{x+t} \, dt \tag{11.38} $$

the form given by (11.36) being merely a convenient approximation to the true value.

These commutation functions apply to insurances payable at the end of the year of death. When the insurance is payable at the moment of death, the payment is made on the average one-half year earlier, and one of the usual approximate relationships may be assumed, for example:

$$\bar{A} \doteq (1 + i)^{\frac{1}{2}} A.$$

In most instances in practice, only a few isolated contingent insurance values are required, and the simplest method of evaluation is approximate integration. This device is particularly convenient when the mortality assumption must be changed at some point in the contract. Suppose, for example, that $\bar{A}^2_{xy}$ is to be evaluated on the basis of mortality from Table A while both lives survive and from Table B after the death of (y). This is easily handled by first using (11.32) to write

$$\bar{A}^2_{xy} = \frac{1}{l_x l_y} \int_0^\infty v^t l_{x+t} l_{y+t} \mu_{y+t} \bar{A}_{x+t} \, dt.$$

Then, after an appropriate formula of approximate integration has been chosen, the required values of l_{x+t}, l_{y+t}, and μ_{y+t} are taken from Table A and those of $\bar{A}_{x+t}$ from Table B.

As in the case of contingent probabilities, an exact evaluation is possible when either Gompertz's or Makeham's law applies. With Gompertz's law, we have

$$\begin{aligned} \bar{A}^1_{x_1 x_2 \cdots x_m} &= {}_\infty q^1_{x_1 x_2 \cdots x_m} \cdot \bar{A}_{x_1 x_2 \cdots x_m} \\ &= \frac{c^{x_1}}{c^w} \bar{A}_w, \text{ where } c^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}. \end{aligned} \tag{11.39}$$

In this case, the value of the contingent insurance is the same as the value of the joint-life insurance multiplied by the probability that (x) will die first.

With Makeham's law, we have

$$\bar{A}^1_{x_1 x_2 \cdots x_m} = \frac{c^{x_1}}{c^w} \left(\frac{1}{m} \bar{A}_{ww \cdots w} + \log s \cdot \bar{a}_{ww \cdots w} \right) - \log s \cdot \bar{a}_{ww \cdots w},$$

$$\text{where } mc^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}. \tag{11.40}$$

The derivations of these formulas are not given here, since they parallel closely those of the corresponding contingent probabilities which were set forth in detail in Section 3.

6. Annual premiums and reserves

A contract of contingent insurance may terminate in either of two ways: (1) by the occurrence of the event insured against, in which case a claim is incurred; or (2) by the occurrence of a death contrary to the order of deaths insured against, in which case the contract expires. When contracts of this type are subject to annual premiums, both of the above possibilities must be considered in determining the premium-paying period, since premiums should not be made payable beyond the period during which the insurance is in force.

In the case of an insurance contingent upon a specified status being the *first* to fail, the contract will either become a claim or expire upon the occurrence of the first failure, and the annual premiums are accordingly made payable during the joint existence of all the statuses. This is illustrated by

$$P^{\overline{1}}_{xy} = \frac{A^{\overline{1}}_{xy}}{\ddot{a}_{xy}}$$

and

$$P^{\overline{1}}_{x:\overline{yz}} = \frac{A^{\overline{1}}_{x:\overline{yz}}}{\ddot{a}_{x:\overline{yz}}}.$$

When an insurance is contingent upon a specified status being the *last* to fail, the insurance continues in force until the failure of the specified status, and the annual premiums may be made payable during the existence of the failing status alone. In the general case involving m statuses where the insurance is contingent upon a specified status being the n-th to fail, the annual premiums may be made payable during the joint existence of the failing status with at least $m-n$ of the other statuses. In practice, however, there are often underwriting considerations which require that premium payments be limited to a period shorter than the maximum indicated above, and consequently it is not possible to give unique definitions for functions like $P^{\overline{2}}_{xyz}$.

Terminal reserves for contingent insurances are computed by the prospective method, and the amount of the reserve at any time depends upon the then surviving combination of lives. As an illustration, consider the terminal reserve at the end of t years for an insurance on (x) if he dies second of the three lives (x), (y), and (z), with annual premiums payable during the joint lifetime of all

three. If the lives are all surviving at the end of t years, the reserve is

$$A^{\frac{2}{}}_{\frac{x+t:y+t:z+t}{}} - P^{2}_{xyz} \cdot \ddot{a}_{x+t:y+t:z+t} .$$

If (y) has died, the reserve is $A^{\frac{1}{}}_{\frac{x+t:z+t}{}}$, and if (z) has died, the reserve is $A^{\frac{1}{}}_{\frac{x+t:y+t}{}}$.

7. Some special contingent functions

The probability that (x) will be alive n years after the death of (y) is

$$\int_0^\infty {}_{t+n}p_x \cdot {}_tp_y \mu_{y+t}\, dt = {}_np_x \int_0^\infty {}_tp_{x+n:y} \mu_{y+t}\, dt = {}_np_x \cdot {}_\infty q^{\frac{1}{}}_{\frac{x+n:y}{}} . \quad (11.41)$$

Similarly the probability that (y) will be alive n years after the death of (x) is ${}_np_y \cdot {}_\infty q^{\frac{1}{}}_{\frac{x:y+n}{}}$, and the probability that at least n years will elapse between the two deaths is therefore

$${}_np_x \cdot {}_\infty q^{\frac{1}{}}_{\frac{x+n:y}{}} + {}_np_y \cdot {}_\infty q^{\frac{1}{}}_{\frac{x:y+n}{}} . \quad (11.42)$$

It follows that the probability that (x) and (y) will die within n years of each other is

$$1 - ({}_np_x \cdot {}_\infty q^{\frac{1}{}}_{\frac{x+n:y}{}} + {}_np_y \cdot {}_\infty q^{\frac{1}{}}_{\frac{x:y+n}{}}) . \quad (11.43)$$

The probability that (x) will die before (y) or within n years after the death of (y) is the complement of the probability that (x) will be alive n years after the death of (y), or

$$1 - {}_np_x \cdot {}_\infty q^{\frac{1}{}}_{\frac{x+n:y}{}} . \quad (11.44)$$

The probability that (x) will die during the n years following the death of (y) is

$$\int_0^\infty ({}_tp_x - {}_{t+n}p_x) {}_tp_y \mu_{y+t}\, dt = {}_\infty q^{\frac{1}{}}_{xy} - {}_np_x \cdot {}_\infty q^{\frac{1}{}}_{\frac{x+n:y}{}} , \quad (11.45)$$

which is evidently correct since the required probability is the probability that (x) will survive (y) but not be alive n years after the death of (y).

We may now consider the values of insurances which depend on contingent probabilities of the above type.

The net single premium for a pure endowment of 1 payable if (x) is alive n years after the death of (y) is

$$\int_0^\infty v^{t+n}\,{}_{t+n}p_x\,{}_tp_y\mu_{y+t}\,dt = \frac{D_{x+n}}{D_x}\,\bar{A}^{\,1}_{x+n:\overline{y}}. \qquad (11.46)$$

The annual premium is obtained by dividing the single premium by the present value of an annuity-due payable until the death of (x) or until n years after the death of (y), whichever period is shorter. The present value is

$$\ddot{a}_{x:\overline{n}|} + \sum_{t=0}^{\infty} v^{n+t}\,{}_{n+t}p_x\,{}_tp_y = \ddot{a}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\sum_{t=0}^{\infty} v^t\,{}_tp_{x+n:y}$$

$$= \ddot{a}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\,\ddot{a}_{x+n:y}. \qquad (11.47)$$

In this expression, the first term represents the present value of the first n payments, which are independent of the survival of (y), and the second term takes account of the payments after n years, which are made only if (y) was alive n years before.

The present value of an insurance payable upon the death of (x) provided it occurs at least n years after the death of (y) is

$$\int_0^\infty v^{n+t}(1 - {}_tp_y)_{n+t}p_x\cdot\mu_{x+n+t}\,dt = \frac{D_{x+n}}{D_x}(\bar{A}_{x+n} - \bar{A}^{\,1}_{\underline{x+n:y}}). \qquad (11.48)$$

The annual premium is determined by dividing by $\ddot{a}_x$.

The present value of an insurance payable if (x) dies before (y) or within n years after (y) is equal to $\bar{A}_x$ minus the value of an insurance payable if (x) dies at least n years after (y):

$$\bar{A}_x - \frac{D_{x+n}}{D_x}(\bar{A}_{x+n} - \bar{A}^{\,1}_{\underline{x+n:y}}). \qquad (11.49a)$$

Alternatively, this value may be expressed as

$$\bar{A}^{\,1}_{x:\overline{n}|} + \int_0^\infty v^{n+t}\,{}_{n+t}p_x\cdot{}_tp_y\cdot\mu_{x+n+t}\,dt$$

since the insurance must be payable if (x) dies during the first n years, and is payable after that time if (y) is alive n years before the death of (x). This alternative expression reduces to

$$\bar{A}^{\,1}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\,\bar{A}^{\,1}_{\underline{x+n:y}}, \qquad (11.49b)$$

which may be seen to be equivalent to the formula first obtained.

The annual premium is determined by dividing by

$$\bar{a}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\,\bar{a}_{x+n:y}\,.$$

8. Summary of notation

1. The notational principle used with simple contingent functions is an extension of Principle I (Chapter 3):

When a suffixed subscript consists of several statuses, a numeral, say r, written above one of the statuses indicates that the function is determined by that status being the r-th to fail. For example, the suffix $x\overset{2}{y}z$ refers to the contingency that (y) will be the second to die of the three lives; the suffix $\overline{xy}:\overset{2}{z}w$ to the contingency that (z) will be the second to fail of the three statuses $(\overline{xy})$, (z), (w).

2. With contingent term insurances, the temporary period is denoted by a prefixed subscript; e.g., ${}_nA^2_{xy}$. (When the insurance is contingent upon the first death, however, either ${}_nA^1_{xy}$ or $A^1_{xy:\overline{n}|}$ is correct.) This device is sometimes used as an alternative notation under other circumstances. For example, the symbol $a_{x:\overline{n}|}$ is sometimes written ${}_na_x$, and $A^1_{x:\overline{n}|}$ is sometimes written ${}_nA_x$.

3. Contingent commutation functions:

$$C^1_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}+1}\int_0^1 l_{x_1+t:x_2+t:\cdots:x_m+t}\mu_{x_1+t}\,dt$$

$$\bar{C}^1_{x_1x_2\cdots x_m} = \int_0^1 D_{x_1+t:x_2+t:\cdots:x_m+t}\mu_{x_1+t}\,dt$$

The functions M, $\bar{M}$, R, $\bar{R}$ are obtained by summation in the usual way.

References

1, 2, 3, 4. This material is also covered by Spurgeon (1932) and by Hooker and Longley-Cook (1957).

5. The method of formula (11.40) is due to Hume and Stott (1905). Another method of evaluating $\bar{A}^1_{x_1x_2\cdots x_m}$ under Makeham's law was given by Evans (1925) and may be seen in Hooker and Longley-Cook (1957), page 147. A paper by Groeschell and Snell (1951) shows how Evans' method is applied to the valuation of reversionary interests for taxation purposes. The numerical illustrations in this paper, comparing the results of Evans' method

with others described in the present chapter, will be of interest to students.

Spurgeon (1932) and Hooker and Longley-Cook (1957) describe several other approximate methods of evaluation. Some of these form the bases for Exercises 5, 6, 19, 20, and 21.

6. The family-type policy currently issued by many insurance companies affords an interesting application of contingent insurance principles. A paper by Sarnoff (1958) discusses this policy, with special reference to the problem of valuation.

EXERCISES

1, 2. Contingent probabilities

1. Show that ${}_{\infty}q^{\overline{2}}_{\overline{xx}:x} = \frac{1}{3}$.

2. Given the three lives A, B, C, all aged x. Write an expression in terms of ${}_np_x$ for the probability that A and B will die within n years, A predeceasing B, and that C will survive for n years.

3. Given ${}_{\infty}q^{2}_{xy} = .4$, ${}_{\infty}q^{2}_{xz} = .5$, and ${}_{\infty}q^{2}_{yz} = .3$, find the numerical value of ${}_{\infty}q^{2}_{\overline{xy}z}$. (Ans. .3)

4. Express ${}_{\infty}q^{2}_{\overline{xx}\overline{yz}}$ in terms of probabilities contingent upon the first death.

3. Evaluation of contingent probabilities

5. Using the fact that $\dfrac{\partial \mathring{e}_{xy}}{\partial x} = \mu_x \mathring{e}_{xy} - {}_{\infty}q^{1}_{xy}$ (see Appendix II), derive the approximate formula

$$ {}_{\infty}q^{1}_{xy} \doteq \mu_x \mathring{e}_{xy} + \tfrac{1}{2}(\mathring{e}_{x-1:y} - \mathring{e}_{x+1:y}). $$

6. (a) Assuming that $\mu_{x+t} \doteq \dfrac{1}{2l_{x+t}}(l_{x+t-1} - l_{x+t+1})$ for all values of t, show that

$$ {}_{\infty}q^{1}_{xy} \doteq \tfrac{1}{2}\left\{\frac{\mathring{e}_{x-1:y}}{p_{x-1}} - p_x \cdot \mathring{e}_{x+1:y}\right\}. $$

(b) Determine whether (x) is older than (y), given

$$ \mathring{e}_{x-1:y} = 10.682 \qquad l_{x-1:y} = 10{,}250{,}000 $$
$$ \mathring{e}_{x+1:y} = 10.000 \qquad l_{x:y} = 10{,}000{,}000 $$
$$ l_{x+1:y} = 9{,}740{,}000. $$

7. In a mortality table known to follow Makeham's law, you are given:

$$ A = .003,\ c^{10} = 3, \text{ and } \mathring{e}_{40:60} = 17. $$

Find ${}_{\infty}q^{1}_{40:20}$. (Ans. .2755)

8. Assuming that Makeham's law holds and given the following data find as closely as possible the value of ${}_{\infty}q^{\overline{1}}_{40:40:40:42}$.

$\log_{10} 4.0 = .60206$

$\log_{10} 4.1 = .61278$ $\qquad c = 1.1$

$\log_{10} 4.2 = .62325$ $\qquad \log_e s = -.00619$

$\log_{10} 4.3 = .63347$ $\qquad \mathring{e}_{40:40:40:40} = 13.880$

$\log_{10} 4.4 = .64345$ $\qquad \mathring{e}_{41:41:41:41} = 13.429$

(Ans. .725)

4. Contingent insurance functions

9. Show that the net single premium for an insurance of 1 payable at the end of the year of death of (x) provided that (y) is then alive may be expressed as

$$vp_y\bar{a}_{x:y+1} - a_{xy}.$$

10. Show by general reasoning that

$$A^1_{xy} - A^2_{xy} > 0.$$

Is this true if $i = 0$? Explain.

11. Show that

$$A^2_{xy} = A_y - A_{xy} + A^1_{xy}.$$

12. Show that

(a) $$\int_0^{\infty} v^t{}_tp_{xy}\mu_{y+t}\,{}_nE_{x+t}\,dt = \frac{D_{x+n}}{D_x}\bar{A}^{\,1}_{x+n:y}$$

(b) $$\int_0^{\infty} v^t{}_tp_{xy}\mu_{y+t}\,{}_{n|}\bar{A}_{x+t}\,dt = \frac{D_{x+n}}{D_x}(\bar{A}_{x+n} - \bar{A}^{\overline{1}}_{x+n:y})$$

In each case, describe the benefit.

13. Express $\bar{A}^2_{wxyz}$ in terms of insurances contingent on the first death.

5. Evaluation of contingent insurances

14. Describe the benefit denoted by the symbol $A^{\overline{1}}_{x:\overline{y:n}|}$ and give a formula for it in commutation symbols.

15. Given $\mu_x = 0.002 + 10^{-6}\,c^x$ and $c^{10} = 3$, express $\bar{A}^1_{40:30}$ in terms of $\bar{A}_{40:30}$ and $\bar{a}_{40:30}$.

16. Give the full derivation of formulas (11.39) and (11.40).

17. Show that, if a mortality table follows Makeham's law,

$$\bar{A}^1_{xy} = \frac{1}{1+c^{y-s}}\bar{A}_{xy} + \frac{1-c^{y-s}}{1+c^{y-s}}\log s \cdot \bar{a}_{xy}.$$

18. Given the following data from a table known to follow Makeham's law, find the value of $\bar{A}^1_{30:40:60}$.

$$\mu_{30} = .0106$$

$$\mu_{40} = .0136$$

$$\mu_{60} = .0316$$

x	c^x	$\bar{a}_{xxx}$	$\bar{A}_{xxx}$
49	80.6	9.0	.73
50	88.2	8.7	.74

(Ans. .099)

19. Show that $\frac{\partial \bar{a}_{xy}}{\partial x} = \mu_x \bar{a}_{xy} - \bar{A}^1_{xy}$. Hence derive the approximate formula

$$\bar{A}^1_{xy} \doteq \mu_x \bar{a}_{xy} + \tfrac{1}{2}(\bar{a}_{x-1:y} - \bar{a}_{x+1:y}).$$

20. Assuming that $\mu_{x+t} \doteq \frac{l_{x+t-1} - l_{x+t+1}}{2l_{x+t}}$

for all values of t, derive the formula

$$\bar{A}^1_{xy} \doteq \tfrac{1}{2}\left\{\frac{\bar{a}_{x-1:y}}{p_{x-1}} - p_x \cdot \bar{a}_{x+1:y}\right\}.$$

What is the corresponding formula for $\bar{A}^1_{\overline{xy}}$?

21. Assuming that $\mu_{x+t} \doteq q_{x+t}$ for all values of t, derive the formula

$$\bar{A}^1_{xy} \doteq \bar{a}_{xy} - p_x \cdot \bar{a}_{x+1:y}.$$

6. Annual premiums and reserves

22. Give formulas for ${}_tV^{\delta}_{xyz}$, assuming that the annual premium is payable during the joint lifetime of (x) and the survivor of (y) and (z).

23. A policy insures a husband aged 40 and his wife aged 35. If (35) dies first, the payment at her death is \$1000. If (40) dies first, the payment at his death is \$10,000 during the first policy year; thereafter it decreases by \$100 at the beginning of each policy year until it reaches \$8000 and then remains constant. Give a formula for the net annual premium in terms of commutation functions.

7. Some special contingent functions

24. In the case of 3 lives aged 70, 75, and 79, the probability that exactly 2 lives survive 5 years is .45 and that at least one dies within 5 years is

.775. Given that $\frac{1}{{}_5p_{70}}$, $\frac{1}{{}_5p_{75}}$, and $\frac{1}{{}_5p_{70}}$ are in arithmetic progression and that ${}_{\infty}q^{1}_{70:80} = .3$, find the probability that 2 lives aged 70 and 75 will die within 5 years of each other. (Ans. .445)

25. Show that the net annual premium for a pure endowment of 1 payable if (x) is alive n years after the death of (y) is given by

$$\frac{\bar{A}^{\,1}_{x+n:y}}{\bar{s}_{x:\overline{n}|} + \bar{a}_{x+n:y}}.$$

26. A deferred annuity of 1 is payable annually, beginning m years from now. Annuity payments are made if at the time of payment (x) is alive and (y) either is alive or has been dead less than n years $(m > n)$. What is the present value of this annuity?

27. Find the present value of 1 payable at the moment of death of (50) provided that (20) has then either died or attained age 40.

28. Derive and simplify an expression, in terms of single premium symbols, for an insurance payable on the death of (x) after (y), provided (y) dies during the n years preceding the death of (x).

Miscellaneous problems

29. Find the single premium with a loading of 7½ per cent of the net premium for a contingent survivorship policy payable if a life aged 30 dies before a life aged 60, or within 5 years after the death of the latter, with return of the single premium without interest 5 years after the death of the life aged 60 if no claim under the policy arises by the death of the life aged 30.

30. Find the value of $\int_0^{\infty} {}_tq^{\,2}_{xz}\ {}_tp_{xz}\ \mu_{x+t}\,dt$.

31. An insurance is to be paid at the moment of death of (x) if he survives (y). The payment is equal to the number of years (including fractional parts of a year) elapsed from the date of issue to the moment of death of (y). Write the net single premium as a definite integral.

32. Develop a formula in terms of an expression which does not involve integrals for the net single premium for an insurance of 1 payable 10 years after the death of (x), if both or either of (y) and (z) survive (x) and both are dead before the end of the 10-year period.

33. A corporation has a president, aged 50, and two vice-presidents aged 48 and 45 respectively. They wish to purchase the following insurance: If the president dies first, $1000 is to be paid immediately and $2000 is to be paid upon the first death of the two vice-presidents. If one of the vice-presidents dies first, nothing is to be paid until the second death among the three officers at which time $3000 is to be paid.

Compute the net single premium for this insurance, assuming immediate

payment of claims, given the following values at 2½% from the mortality table to be employed:

$$\bar{a}_{50:48} = 10.95 \qquad \bar{a}_{48:45} = 11.46$$

$$\bar{a}_{50:45} = 11.26 \qquad \bar{a}_{50:48:45} = 9.30$$

(Ans. $1974.38)

Chapter 12

COMPOUND CONTINGENT FUNCTIONS

1. Introduction

The contingent functions considered in the preceding chapter were of a type in which the sequence of deaths was restricted by a condition on a single one of the lives (or statuses) involved. Thus a typical problem arose from the condition that (y) die second of the lives (x), (y), and (z), and nothing was said about the multiple condition exemplified by, say, the requirement that (y) die second predeceased by (x) among (x), (y), and (z). The present chapter deals with functions of this latter type, known as *compound contingent functions*.

It is obvious that no compound contingent function can arise from a two-life status, since an ordering restriction on one life will automatically determine a complementary condition on the other. A compound contingent function thus requires three or more lives with two or more conditions on the sequence of deaths.

2. Probabilities

Given the three lives (x), (y), and (z), what is the probability that their deaths will occur in the order named? This probability is denoted by ${}_{\infty}q^{\,2}_{\,1\;xyz}$, and it may be expressed in several different ways:

$$ {}_{\infty}q^{\,2}_{\,1\;xyz} = \int_0^{\infty} {}_tp_{xyz}\mu_{z+t} \cdot {}_{\infty}q^{\,1}_{\,1\;y+t:z+t}\, dt \qquad (12.1a) $$

or

$$ \int_0^{\infty} {}_tq_x \cdot {}_tp_{yz}\mu_{y+t}\, dt \qquad (12.1b) $$

or

$$ \int_0^{\infty} {}_tq^{\,2}_{xy} \cdot {}_tp_z\mu_{z+t}\, dt. \qquad (12.1c) $$

These three expressions are derived by letting t refer to the moment of death of (x), of (y), and of (z) respectively. The integrand in the first case expresses the condition that at the moment of death of (x), the lives (y) and (z) are both surviving and that (y) will later predecease (z). The second integral reflects the fact that when (y)

dies, (x) is already dead and (z) is surviving. The third expression is based on the fact that (x) and (y) have already died in the order named when the death of (z) occurs.

The three integrals may easily be shown to be equivalent mathematically. If, in (12.1b), the substitution of $\int_0^t {}_sp_z\mu_{x+s}\,ds$ is made for ${}_tq_z$, a double integral results. By inverting the order of integration and simplifying, formula (12.1a) is obtained. If, on the other hand, the substitution of $\int_t^\infty {}_sp_z\mu_{z+s}\,ds$ is made for ${}_tp_z$ in formula (12.1b), then when the order of integration is inverted formula (12.1c) will be obtained.

All three expressions will normally require approximate integration methods, and formulas (12.1a) and (12.1c) are less convenient to evaluate because of the presence of contingent functions under the integral sign.

By using (12.1b), it is possible to express ${}_\infty q^{\ 2}_{xyz}$ in terms of simple contingent probabilities. Writing $1 - {}_tp_x$ for ${}_tq_x$, we have

$$\begin{aligned} {}_\infty q^{\ 2}_{xyz} &= \int_0^\infty (1 - {}_tp_x)\,{}_tp_{yz}\mu_{y+t}\,dt \\ &= \int_0^\infty {}_tp_{yz}\mu_{y+t}\,dt - \int_0^\infty {}_tp_{xyz}\mu_{y+t}\,dt \\ &= {}_\infty q^{\ 1}_{yz} - {}_\infty q^{\ 1}_{xyz}. \end{aligned} \tag{12.2}$$

In the symbols used for compound contingent functions, it is important to note the significance of the numerals which appear above and below the suffix. The numeral above the suffix distinguishes the life upon whose failure the function is finally determined, and numerals below the suffix indicate the order in which the other lives are to fail. In the case of ${}_\infty q^{\ 2}_{xyz}$, for example, the probability is determined upon (y)'s dying second, predeceased by (x).

When more than three lives are involved, the functions become more complex and it is not always possible to set up integrals which are themselves free of contingent functions. For example, to take a four-life case, the probability that (w), (x), (y), and (z) will die in the order named is

$$ {}_{\infty}q^{\ 3}_{wxyz\ 12} = \int_0^{\infty} {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t} \cdot {}_{\infty}q^{\ 1}_{y+t:z+t}\, dt \qquad (12.2a) $$

or

$$ \int_0^{\infty} {}_tq^{\ 2}_{wx} \cdot {}_tp_{yz}\mu_{y+t}\, dt. \qquad (12.2b) $$

Other equivalent expressions can be formulated by basing the integrals on the moment of death of (w) or of (z). It will be found, however, that the formulas are simpler when the variable of integration t refers to the moment of death of one of the intermediate lives, as (x) in (12.2a) or (y) in (12.2b).

3. Insurances

A single order of deaths among a group of lives may give rise to several different insurance functions. If there are three lives which are to die in the order (x), (y), (z), there are three insurance functions corresponding to payment of the benefit upon the death of (x), of (y), or of (z). Actually, the first case presents nothing new since if the insurance is payable upon (x)'s dying first, the order of the subsequent deaths is immaterial. There remain the two cases where the insurance is payable (1) upon the death of (y) predeceased by (x) and survived by (z), and (2) upon the death of (z) predeceased by (x) and (y) in that order.

For case (1), where the payment is made upon the death of (y), the net single premium for an insurance of 1 is given by

$$ \begin{aligned} \bar{A}^{\ 2}_{1\ xyz} &= \int_0^{\infty} v^t \cdot {}_tq_x \cdot {}_tp_{yz}\mu_{y+t}\, dt \\ &= \int_0^{\infty} v^t(1 - {}_tp_x){}_tp_{yz}\mu_{y+t}\, dt \\ &= \bar{A}^{1}_{yz} - \bar{A}^{\ 1}_{1\ xyz}. \end{aligned} \qquad (12.3) $$

For case (2), where the payment is made upon the death of (z), we have

$$ \bar{A}^{\ 3}_{1\ xyz} = \int_0^{\infty} v^t \cdot {}_tq_x \cdot {}_tp_{yz}\mu_{y+t}\bar{A}_{z+t}\, dt. \qquad (12.4) $$

Analogous functions may be written for four lives. The student should describe the insurance represented by each of the following:

$$\bar{A}^{\,2}_{\substack{wxyz\\1}} = \int_0^\infty v^t \, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t} \, dt \tag{12.5}$$

$$\bar{A}^{\,3}_{\substack{wxyz\\12}} = \int_0^\infty v^t \, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\bar{A}^{\,1}_{\overline{y+t}:z+t} \, dt \tag{12.6}$$

$$\bar{A}^{\,4}_{\substack{wxyz\\12}} = \int_0^\infty v^t \, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\bar{A}^{\,2}_{y+t:\overline{z+t}} \, dt \tag{12.7}$$

$$\bar{A}^{\,3:4}_{\substack{wx\,y\,z\\12}} = \int_0^\infty v^t \, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\bar{A}_{y+t} \, dt \tag{12.8}$$

Each of the above may also be written as a double integral. For example,

$$\bar{A}^{\,3}_{\substack{wxyz\\12}} = \int_0^\infty {}_tq_w \cdot {}_tp_x\mu_{x+t} \left(\int_t^\infty v^s \, {}_sp_{yz}\mu_{y+s} \, ds \right) dt.$$

In more complex cases, multiple integration is sometimes the only means of writing a precise expression for a given function.

4. Compound contingencies with terms-certain

A compound contingent function may involve conditions other than the mere order of deaths, as for example where a condition as to the time interval between deaths is introduced. No new theory is needed for the formulation of such functions, and in fact they are always best handled from first principles without any attempt at classification. Since these functions are of such varying types, it is not feasible to assign specific symbols for them. The examples which follow will illustrate the techniques used in formulating expressions for such functions.

Consider the probability that (x), (y), and (z) will die in the order named within twenty years, at least five years separating each death. These conditions require the death of (y) within the period from five to fifteen years from the present, and the probability may be expressed as

$$\int_5^{15} (1 - {}_{t-5}p_x) \, {}_tp_y\mu_{y+t}({}_{t+5}p_z - {}_{20}p_z) \, dt.$$

This may be reduced to a form involving simple contingent probabilities. Making the change of variable $s = t - 5$, we have

$$\int_0^{10} (1 - {}_sp_x)_{s+5}p_y\mu_{y+s+5}({}_{s+10}p_z - {}_{20}p_z)\,ds$$

$$= {}_5p_y \cdot {}_{10}p_z \int_0^{10} (1 - {}_sp_x)_sp_{y+5}\mu_{y+5+s} \cdot {}_sp_{z+10}\,ds$$

$$- {}_5p_y \cdot {}_{20}p_z \int_0^{10} (1 - {}_sp_x)_sp_{y+5}\mu_{y+5+s}\,ds$$

$$= {}_5p_y \cdot {}_{10}p_z({}_{10}q^{\frac{1}{y+5}}_{z:z+10} - {}_{10}q^{\frac{1}{x:y+5}}_{z:z+10}) - {}_5p_y \cdot {}_{20}p_z({}_{10}q^{\frac{1}{y+5}}_{y+5} - {}_{10}q^{\frac{1}{x:y+5}}_{x:y+5}).$$

Consider next the probability that (x), (y), and (z) will die in the order named, (y) within five years of (x), and (z) within ten years of (y). This probability may be expressed as

$$\int_0^5 (1 - {}_tp_x)_tp_y\mu_{y+t}({}_tp_z - {}_{t+10}p_z)\,dt$$

$$+ \int_5^\infty ({}_{t-5}p_x - {}_tp_x)_tp_y\mu_{y+t}({}_tp_z - {}_{t+10}p_z)\,dt,$$

since, if (y) dies within the first five years, he must be predeceased by (x), and if (y) dies after five years, the prior death of (x) must have occurred within five years, the death of (z) occurring within ten years after that of (y) in either case. The student may verify that the integrals reduce to the following expression involving simple contingent functions:

$${}_5q^{\frac{1}{yz}}_{xyz} - {}_\infty q^{\frac{1}{xyz}}_{xyz} - {}_{10}p_z({}_5q^{\frac{1}{y:z+10}}_{y:z+10} - {}_\infty q^{\frac{1}{xy:z+10}}_{xy:z+10})$$

$$+ {}_5p_{yz}({}_\infty q^{\frac{1}{x:y+5}}_{x:z+5} - {}_{10}p_{z+5} \cdot {}_\infty q^{\frac{1}{x:y+5}}_{x:z+15}).$$

Suppose that we require the probability that the three lives (10), (20), (30) will all die between ages 40 and 80 and in the order named, the first and second deaths occurring within five years of each other, with the second and third deaths separated by at least five years. It may be noted that, in order to fulfill these conditions, the life (20) must die within the period from thirty to forty-five years from the present, and that two separate integrals must be written to take account of (10)'s predeceasing (20) within five years. We thus have

$$\int_{30}^{35} ({}_{30}p_{10} - {}_tp_{10})_tp_{20}\mu_{20+t}({}_{t+5}p_{30} - {}_{50}p_{30})\,dt$$

$$+ \int_{35}^{45} ({}_{t-5}p_{10} - {}_tp_{10})_tp_{20}\mu_{20+t}({}_{t+5}p_{30} - {}_{50}p_{30})\,dt.$$

As an example involving four lives, consider the probability that (w), (x), (y), and (z) will die in the order named, the first two deaths occurring within the first five years with not more than two years separating each of the last three deaths. This probability requires a double integral:

$$\int_0^5 (1 - {}_tp_w)\,{}_tp_x\mu_{x+t}\left(\int_t^{t+2} {}_sp_y\mu_{y+s}({}_sp_z - {}_{s+2}p_z)\,ds\right)dt.$$

Insurances which involve contingencies of this nature may be similarly formulated. Suppose that an insurance of 1 is payable at the death of (x) if he dies before (y) or within n years after (y), leaving a third life (z) surviving. In expressing the net single premium, we note that during the first n years the insurance is independent of the survival of (y), and write

$$\int_0^n v^t\,{}_tp_{xz}\mu_{x+t}\,dt + v^n{}_np_{xz}\int_0^\infty v^t\,{}_tp_{x+n:y:z+n}\mu_{x+n+t}\,dt$$

$$= \bar{A}^{1}_{xz:\overline{n}|} + \frac{D_{x+n:z+n}}{D_{xz}}\cdot\bar{A}_{\frac{1}{x+n:y:z+n}}.$$

As a second example, consider an insurance payable at the death of (x) if he is predeceased by (y) and (z) in the order named with the death of (z) occurring at least n years earlier. This may be regarded as a contingent insurance on the life of (z) in which the amount of insurance at any time is equal to the present value at that time of an n-year deferred insurance on (x). The value may thus be written as

$$\int_0^\infty v^t\,{}_tq_y\,{}_tp_{xz}\mu_{x+t}\cdot{}_n|\bar{A}_{x+t}\,dt$$

$$= \frac{D_{x+n}}{D_x}\int_0^\infty v^t\,{}_tq_y\,{}_tp_{x+n:z}\mu_{x+t}\cdot\bar{A}_{x+n+t}\,dt = \frac{D_{x+n}}{D_x}\cdot\bar{A}_{\frac{2}{x+n:y:z}}.$$

5. Summary of notation

Numerals are used with a suffixed subscript to indicate the order of failure of the statuses involved. The numeral written above the suffix distinguishes the status upon whose failure the event is finally determined, and the numerals written below the suffix indicate the order in which the other statuses are to fail. In cases where a numeral below the suffix is clearly implied, it is unnecessary

to write it, as, for example, the numeral 2 below the y in $\bar{A}^{3}_{1\,xyz}$, or the numeral 3 below the y in $\bar{A}^{2}_{1\,\overline{wx:yz}}$.

References

This material is treated by Spurgeon (1932), Chapter 19, and more briefly by Hooker and Longley-Cook (1957), Chapter 24.

EXERCISES

1, 2. Probabilities

1. Express ${}_{\omega}q^{2}_{1\,wxyz}$

   (a) as a definite integral;
   (b) in terms of simple contingent probabilities.

2. If the mortality table follows Gompertz's law, show that

   (a) ${}_{\omega}q^{2}_{1\,xyz} = {}_{\omega}q^{1}_{yz} \cdot {}_{\omega}q^{1}_{xyz}$

   (b) ${}_{\omega}q^{3}_{1\ 2\,25:30:35:40} = \dfrac{1}{1+c^{5}} \cdot {}_{\omega}q^{2}_{1\,25:30:35:40}$.

3. Show that

$${}_{\omega}q^{1}_{w:\overline{xyz}} = {}_{\omega}q^{2}_{wxyz} + {}_{\omega}q^{1}_{wxy} - {}_{\omega}q^{2}_{wxy} - {}_{\omega}q^{2}_{wz} - {}_{\omega}q^{1}_{\overline{wx};y} - 2({}_{\omega}q^{2}_{1\,wxyz} + {}_{\omega}q^{2}_{1\,wxyz} - 1).$$

4. Given

$${}_{\omega}q^{1}_{xy} = .5537,\ {}_{\omega}q^{1}_{zz} = .6484,\ {}_{\omega}q^{1}_{zyz} = .5325,\ {}_{\omega}q^{2}_{zyz} = {}_{\omega}q^{3}_{xyz}.\ \text{Find } {}_{\omega}q^{2}_{1\,xyz}.$$

(Ans. .2145)

3. Insurances

5. Show that

   (a) $\bar{A}^{3}_{1\,xyz} = \bar{A}^{2}_{yz} - \bar{A}^{2:3}_{1\,xyz}$

   (b) $\bar{A}^{2}_{1\,wxyz} = \bar{A}^{1}_{xyz} - \bar{A}^{1}_{wxyz}$.

6. The expressions below have been given for the value of $\bar{A}^{4}_{1\,wxyz}$. Show clearly why each expression is correct or incorrect. Also give another expression, which is correct.

(a) $\int_0^\infty v^t\, {}_tp_w(1 - {}_tp_s)[{}_tp_y(1 - {}_tp_z)\mu_{y+t} + {}_tp_z(1 - {}_tp_y)\mu_{z+t}]\bar{A}_{w+t}\, dt$

(b) $\int_0^\infty v^t\, {}_tp_w \left({}_tp_z \cdot {}_tq_{yz}^{\overline{2}}\bar{A}_{\overline{w+t}:z+t}^{\overline{2}} + {}_tp_y \cdot {}_tq_{zz}^{\overline{2}}\bar{A}_{\overline{w+t}:y+t}^{\overline{2}}\right) dt.$

7. Assume that the mortality of (y) and (z) is given in terms of a standard table and that (x) is subject to the mortality of a special table in which the force of mortality is twice that of the standard table. On the basis of this assumption, derive an expression in terms of the standard table for the value of a unit payable to (z) upon the death of (x), if (z) is then living, provided (x) dies after (y).

8. What annuity value should be used in obtaining the net annual premium corresponding to each of the following net single premiums?

(a) $A_{\underset{1}{xyz}}^{\overline{2}}$

(b) $A_{\underset{1}{wxyz}}^{\overline{2}}$

4. Compound contingencies with terms-certain

9. Express in the form of integrals the probabilities that of the four lives (w), (x), (y), and (z),

(a) the deaths will occur in the order (w), (x), (y), (z), and that (x) and (y) will die within four years of each other;

(b) (w) only will die within 10 years of the present time and (x), (y), and (z) will die after the 10 years in that order, (y) within 5 years of (x) and (z) within 5 years of (y).

10. Express in integral form:

(a) The probability that three lives aged 10, 20, and 30 all die before attaining age 60, with (20) being the second to die;

(b) the probability that (w), (x), (y), and (z) will die in the order named, not more than two years separating any two consecutive deaths.

11. Derive an expression in terms of contingent insurances and commutation functions for the net single premium for an insurance payable at the death of a person now aged 35 if he dies before the survivor of two persons now aged 40 and 50 or within ten years after the death of such survivor, leaving another person now aged 30 surviving.

Miscellaneous problems

12. According to a certain mortality table, the probability that three lives aged 70, 55, and 40 will die in that order at intervals of not less than fifteen years is .048, and the probability that at least one of two lives now aged 70 will be alive fifteen years before the death of a life now aged 55 is .8. Find the probability that neither of two lives now aged 40 will survive to age 70. (Ans. .2704)

13. If it is assumed that $\mu_{x+t} = q_{x+t}$ for all values of t, show that

$$\bar{A}^{\,1}_{\underset{1}{xys}} = \frac{1}{\delta}\,[\bar{A}_{xys} - \bar{A}_{xs} - (1 - \mu_x)(\bar{A}_{x+1:ys} - \bar{A}_{x+1:s})].$$

14. Write an expression in the form of integrals for the gross single premium, including a loading of 10 per cent of the net, for an insurance payable on the death of a life aged 30, provided that he dies either before a life aged 35 or within three years after his death and that both are predeceased by two lives each aged 40 who must die in a specified order. Should the life aged 30 die within the three years after the death of the life aged 35, the sum insured is to be reduced by 25 per cent, but should he survive the life aged 35 by three years, the other lives dying as before, the premium is to be returned.

15. The will of H, now dead, leaves to his wife (w) the income for life from a \$200,000 trust fund, the principal to be divided at (w)'s death in accordance with the following conditions. If H's sons (x) and (y) are both alive, they alone share the principal. If either son predeceases (w), his children living at (w)'s death share the principal. (x) has three children, (a), (b), and (c); and (y) two, (d) and (e). All beneficiaries at (w)'s death are to share the principal per capita, i.e., equally without regard to generation. Ignore the possibility of the birth of any further children to (x) or (y).

Find an expression for the present value of (x)'s expectation (not including that of his children) in terms of continuous contingent insurances payable on the first death.

Chapter 13

REVERSIONARY ANNUITIES

1. Introduction

A reversionary annuity is an annuity which commences upon the failure of a given status (A) if a second status (B) is then in existence and which continues thereafter during the existence of (B). A simple example is the familiar deferred annuity ${}_{n|}a_x$, in which $(\overline{n}|)$ is the failing status (A) and (x) the surviving status (B). In the general case where the failing status involves life contingencies, reversionary annuities are a type of contingent function analogous to the contingent insurances discussed in the preceding chapters.

2. The basic function

The simplest form of reversionary annuity where the failing status involves a life contingency is an annuity of 1 payable to a life now aged x commencing at the end of the year of death of (y)—or, more briefly, an annuity to (x) after (y). The present value of this annuity is denoted by $a_{y|x}$. Since the annuity consists of a series of payments made at the end of each year provided that (x) is alive and (y) is not alive, its present value may be written

$$a_{y|x} = \sum_{t=1}^{\infty} v^t \, {}_tp_x(1 - {}_tp_y) = a_x - a_{xy} . \tag{13.1}$$

This result is obviously correct since this reversionary annuity is the same as a life annuity to (x) with payments omitted during the joint lifetime of (x) and (y). The same reasoning applies to the generalized function $a_{v|u}$, whatever age status or combination of statuses may be represented by (u) and (v), and we may write

$$a_{v|u} = a_u - a_{uv} . \tag{13.2}$$

For example, if (u) represents the joint-life status (xy) and (v) the single life (z), we have

$$a_{z|xy} = a_{xy} - a_{xyz} .$$

Reversionary annuities of this type may thus be expressed directly in terms of single-life and joint-life annuity values. Two further examples follow:

$$a_{\overline{yz}|x} = a_x - a_{x:\overline{yz}} = a_x - a_{xy} - a_{xz} + a_{xyz}$$

$$\begin{aligned} a_{yz|\overline{wx}} &= a_{\overline{wx}} - a_{\overline{wx}:yz} \\ &= a_w + a_x - a_{wx} - a_{wyz} - a_{xyz} + a_{wxyz} . \end{aligned}$$

In the first of these, the annuity is payable during the lifetime of (x) after the death of the survivor of (y) and (z). In the second, the annuity commences upon the first death of (y) and (z) and is payable until the second death of (w) and (x).

Although known as reversionary *annuities*, these functions are essentially insurances since the payments commence upon the failure of a specified status.

3. Modifications of the basic function

The basic function $a_{y|x}$ makes provision for annual payments, and the payment periods are measured from the end of the year of death of (y). In practice, the payments are often made more frequently than once a year, and the payment periods are measured from different points of time.

If the annuity is payable to (x) m times a year and commences at the end of the m-th part of the year in which (y) dies, the present value is given by

$$\begin{aligned} a^{(m)}_{y|x} &= a^{(m)}_x - a^{(m)}_{xy} \\ &\doteqdot a_x - a_{xy} + \frac{m^2 - 1}{12m^2}\mu_y , \end{aligned} \tag{13.3}$$

using (2.18) and recalling that

$$\mu_{xy} = \mu_x + \mu_y .$$

If m becomes infinite in the formula above, we have

$$\bar{a}_{y|x} = \bar{a}_x - \bar{a}_{xy} \doteqdot a_x - a_{xy} + \tfrac{1}{12}\mu_y . \tag{13.4}$$

This represents the present value of an annuity of 1 commencing at the moment of death of (y) and payable continuously throughout the remaining lifetime of (x). An integral expression for this function is

$$\bar{a}_{y|x} = \int_0^\infty v^t \, {}_tp_x (1 - {}_tp_y) \, dt. \tag{13.5a}$$

When the substitution of $\int_0^t {}_sp_y\mu_{y+s} \, ds$ is made for $1 - {}_tp_y$ in (13.5a), we obtain

$$\bar{a}_{y|x} = \int_0^\infty \int_0^t v^t \, {}_tp_x \, {}_sp_y\mu_{y+s} \, ds \, dt.$$

Then, inverting the order of integration,

$$\begin{aligned} \bar{a}_{y|x} &= \int_0^\infty \int_s^\infty v^t \, {}_tp_x \, {}_sp_y\mu_{y+s} \, dt \, ds \\ &= \int_0^\infty {}_sp_y\mu_{y+s} \cdot {}_s|\bar{a}_x \, ds \\ &= \int_0^\infty v^s \, {}_sp_{xy}\mu_{y+s}\bar{a}_{x+s} \, ds. \end{aligned} \tag{13.5b}$$

In this form, the contingent insurance character of the function is clearly apparent.

The symbol $\hat{a}$ is used to distinguish a type of reversionary annuity in which the payments are made periodically with the payment periods measured from the date of death. The annuity which provides payments m times a year to (x) with the first payment of $1/m$ due at the end of $1/m$ years from the moment of death of (y) has the present value:

$$\begin{aligned} \hat{a}^{(m)}_{y|x} &= \int_0^\infty v^t \, {}_tp_{xy}\mu_{y+t}a^{(m)}_{x+t} \, dt \\ &\doteq \int_0^\infty v^t \, {}_tp_{xy}\mu_{y+t}\left(\bar{a}_{x+t} - \frac{1}{2m}\right) dt \\ &\qquad \text{using (2.19) and (2.27),} \\ &= \bar{a}_{y|x} - \frac{1}{2m}\bar{A}^{\,1}_{xy} \\ &\doteq a_x - a_{xy} + \tfrac{1}{12}\mu_y - \frac{1}{2m}\bar{A}^{\,1}_{xy}. \end{aligned} \tag{13.6}$$

It will be observed in each of the three cases considered above that the formula for the modified function consists of the expression

$a_x - a_{xy}$ plus certain correction terms. In practice, the correction terms are often ignored, and the net single premium for all three types is computed as $a_x - a_{xy}$.

This convenient approximation may also be extended to the general case of an annuity to (u) after (v):

$$a^{(m)}_{v|u} \doteqdot \bar{a}_{v|u} \doteqdot \grave{a}^{(m)}_{v|u} \doteqdot a_{v|u} = a_u - a_{uv}.$$

When necessary, the approximation can be improved by adding the proper correction terms. In particular, when either of the statuses (u) and (v) involves a term-certain, the correction terms are important and should not be neglected. This case will be examined in the next section.

From the relation $a_{v|u} \doteqdot \bar{a}_{v|u}$, it follows that annuities with periodic payments may be evaluated as continuous annuities and that either type may therefore be directly computed by methods of numerical integration in cases where the necessary joint-life functions are not already tabulated. As long as (u) and (v) involve only lives, the definite integrals always assume the following forms:

$$\bar{a}_{v|u} = \int_0^\infty v^t \, {}_tp_u(1 - {}_tp_v)\, dt \tag{13.7a}$$

or

$$\int_0^\infty v^t \, {}_tp_{uv}\mu_{v+t}\bar{a}_{u+t}\, dt. \tag{13.7b}$$

The second form is useful when there is a change in the interest or mortality assumptions after the failure of (v).

4. Functions involving terms-certain

We have already noted that the basic formula

$$a_{v|u} = a_u - a_{uv}$$

is valid regardless of the type of status represented by (u) and (v). Let us consider a few cases where these statuses include a term-certain.

If (v) is the term-certain $(\overline{n}|)$ and (u) the single life (x), we have

$$a_{\overline{n}||x} = a_x - a_{x:\overline{n}|},$$

and this will be recognized as the single-life deferred annuity, ${}_{n|}a_x$, cited earlier as a simple example of a reversionary function.

An annuity to (x) after (y) with no payments to be made after n years from the present time has the present value

$$a_{y|x:\overline{n}|} = a_{x:\overline{n}|} - a_{xy:\overline{n}|}.$$

This function may be described as an n-year temporary reversionary annuity. An alternative notation is ${}_{n}a_{y|x}$.

An annuity to (x) after (y) with no payments to be made within the first n years from the present time may be interpreted as an annuity to (x) after $(\overline{y:n}|)$, and

$$\begin{aligned} a_{\overline{y:n}|x} &= a_x - a_{x:\overline{y:n}|} = a_x - a_{xy} - a_{x:\overline{n}|} + a_{xy:\overline{n}|} \\ &= {}_{n|}a_x - {}_{n|}a_{xy}. \end{aligned}$$

The function is an n-year deferred reversionary annuity. An alternative symbol for the present value is ${}_{n|}a_{y|x}$.

It should be noted that the term-certain in the above functions is measured from the date of issue of the contract. It is important to distinguish clearly another type of function in which the temporary or deferred period is measured from the *date of death* instead of the date of issue. In the case where the deferred period is measured from the end of the year of death of (y), the benefit is an annuity to (x) commencing n years after the death of (y), and the present value is

$$\sum_{t=1}^{\infty} v^{n+t}\, {}_{n+t}p_x(1 - {}_{t}p_y) = \frac{D_{x+n}}{D_x}(a_{x+n} - a_{x+n:y}) = \frac{D_{x+n}}{D_x}\, a_{y|x+n}.$$

For the case where the annuity is payable to (x) for n years after the death of (y), the present value is obtained by deducting the value of the above deferred benefit from the normal reversionary annuity:

$$a_{y|x} - \frac{D_{x+n}}{D_x}\, a_{y|x+n}.$$

Although the approximation $a^{(m)}_{v|u} \doteq a_{v|u}$ is often used when (u) and (v) involve only lives, it should not be used when terms-certain are involved. This may be seen in the case of the simple function $a^{(m)}_{\overline{n}|x}$:

$$\begin{aligned} a^{(m)}_{\overline{n}|x} &= a^{(m)}_{x} - a^{(m)}_{x:\overline{n}|} \\ &\doteq \left(a_x + \frac{m-1}{2m}\right) - \left[a_{x:\overline{n}|} + \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right)\right] \\ &= a_x - a_{x:\overline{n}|} + \frac{m-1}{2m} \cdot \frac{D_{x+n}}{D_x}. \end{aligned} \tag{13.8}$$

The term $\frac{m-1}{2m} \cdot \frac{D_{x+n}}{D_x}$ may make an appreciable contribution to the total value of the function and should not be neglected.

5. Annual premiums and reserves

When the reversionary annuity $a_{v|u}$ is subject to annual premiums, the premium-paying period coincides with the joint lifetime of the statuses (u) and (v). If (v) is the first to fail, the annuity becomes payable, whereas if (u) is the first to fail, the contract expires, and in either case no further premium payments are made. The following annual premium formulas illustrate this principle:

$$P(a_{y|x}) = \frac{a_{y|x}}{\ddot{a}_{xy}} = \frac{a_x - a_{xy}}{\ddot{a}_{xy}} = \frac{\ddot{a}_x - \ddot{a}_{xy}}{\ddot{a}_{xy}} = \frac{\ddot{a}_x}{\ddot{a}_{xy}} - 1 \qquad (13.9)$$

$$P(a_{\overline{yz}|x}) = \frac{a_{\overline{yz}|x}}{\ddot{a}_{xyz}}$$

$$P(a_{\overline{yz}|x}) = \frac{a_{\overline{yz}|x}}{\ddot{a}_{x:\overline{yz}}}$$

Reserve formulas are easily written in prospective form. For example,

$${}_tV(a_{y|x}) = a_{y+t|x+t} - P \cdot \ddot{a}_{x+t:y+t} \qquad (13.10)$$

if both (x) and (y) are surviving, or ${}_tV(a_{y|x}) = a_{x+t}$ if (x) alone is surviving. Reversionary annuities often give rise to negative reserves—depending on the ages involved, the duration, and the mortality table used. This feature may be avoided by suitably limiting the premium-paying period.

6. Continuous instalments

Reversionary annuities are sometimes issued in combination with other insurance benefits. One such combination policy is known as *insurance with continuous instalments.*

Suppose that a policy provides that at the death of (x) a monthly annuity of 1 per annum will be payable for n years certain and for as long thereafter as a beneficiary, aged y at date of issue, survives. At the date of issue, the certain payments have a present value of $\ddot{a}^{(12)}_{\overline{n}|} \cdot \bar{A}_x$, and the deferred annuity to (y), the continuous instalment portion, has the following value:

$$\int_0^\infty v^t {}_tp_{xy}\mu_{x+t} \cdot {}_n|\ddot{a}^{(12)}_{y+t}\, dt = \frac{D_{y+n}}{D_y} \int_0^\infty v^t {}_tp_{x:y+n}\mu_{x+t}\ddot{a}^{(12)}_{y+n+t}\, dt. \qquad (13.11)$$

The expressions assume that the first monthly income payment is made on the date of death of (x). Earlier in the chapter the symbol $\hat{a}$ was used to indicate that an immediate annuity commences at the moment of death, and if we now use $\hat{\bar{a}}$ to indicate that an annuity-due becomes payable at death, the definite integral in (13.11) may be written as $\hat{\bar{a}}^{(12)}_{x|y+n}$. The total net single premium, including the certain payments, will then be

$$\ddot{a}^{(12)}_{\overline{n}|}\bar{A}_x + \frac{D_{y+n}}{D_y}\hat{\bar{a}}^{(12)}_{x|y+n}.$$

This particular combination of benefits is known as *whole life insurance with continuous instalments.*

The continuous instalment portion of this policy expires without value if (y) predeceases (x), and when the policy is subject to annual premiums, the payment of the extra premium for the continuous instalment portion is made to depend upon the survival of both (x) and (y). The total annual premium is thus given by

$$\frac{\ddot{a}^{(12)}_{\overline{n}|}\bar{A}_x}{\ddot{a}_x} + \frac{\dfrac{D_{y+n}}{D_y}\hat{\bar{a}}^{(12)}_{x|y+n}}{\ddot{a}_{xy}}, \tag{13.12}$$

reducing to $\dfrac{\ddot{a}^{(12)}_{\overline{n}|}\bar{A}_x}{\ddot{a}_x}$ upon the death of (y).

Continuous instalments may also be issued in combination with other forms of insurance. In the case of *term insurance with continuous instalments,* the benefit is payable only if the insured life (x) dies within m years. The net single premium for the n-year annuity-certain is $\ddot{a}^{(12)}_{\overline{n}|}\bar{A}^{1}_{x:\overline{m}|}$, and the extra premium required to continue the instalments to (y) is

$$\int_0^m v^t {}_tp_{xy}\mu_{x+t:\overline{n}|}\ddot{a}^{(12)}_{y+t}\,dt$$

$$= \int_0^\infty v^t {}_tp_{xy}\mu_{x+t:\overline{n}|}\ddot{a}^{(12)}_{y+t}\,dt - v^m {}_mp_{xy}\int_0^\infty v^t {}_tp_{x+m:y+m}\mu_{x+m+t:\overline{n}|}\ddot{a}^{(12)}_{y+m+t}\,dt$$

$$= \frac{D_{y+n}}{D_y}\hat{\bar{a}}^{(12)}_{x|y+n} - v^m {}_mp_{xy}\frac{D_{y+m+n}}{D_{y+m}}\hat{\bar{a}}^{(12)}_{x+m|y+m+n}$$

$$= \frac{D_{y+n}}{D_y}\left[\hat{\bar{a}}^{(12)}_{x|y+n} - \frac{D_{x+m:y+n+m}}{D_{x:y+n}}\hat{\bar{a}}^{(12)}_{x+m|y+n+m}\right].$$

The total annual premium is

$$\frac{\ddot{a}^{(12)}_{\overline{n}|}\bar{A}^{1}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} + \frac{\dfrac{D_{y+n}}{D_y}\left(\hat{a}^{(12)}_{x|y+n} - \dfrac{D_{x+m:y+n+m}}{D_{x:y+n}}\,\hat{a}^{(12)}_{x+m|y+n+m}\right)}{\ddot{a}_{xy:\overline{m}|}},$$

reducing to $\dfrac{\ddot{a}^{(12)}_{\overline{n}|}\bar{A}^{1}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}}$ upon the death of (y).

In practice, the $\hat{a}^{(12)}$ functions appearing in these formulas are most conveniently evaluated as simple reversionary annuities, thus:

$$\hat{a}^{(12)}_{v|u} \doteq a_u - a_{uv}\,.$$

This approximation understates the true value, a fact which is sometimes reflected in the loading formula.

Reserves for continuous instalment policies may be obtained from the usual valuation principles. As an example, consider an ordinary life policy issued at age x with continuous instalments to a beneficiary aged y. The reserve at the end of t years is given by the following expression:

$$\ddot{a}^{(12)}_{\overline{n}|}\,{}_tV(\bar{A}_x) + \frac{D_{y+n+t}}{D_{y+t}}\,\hat{a}^{(12)}_{x+t|y+n+t} - P\cdot\ddot{a}_{x+t:y+t}\,,$$

where $P = \dfrac{\dfrac{D_{y+n}}{D_y}\,\hat{a}^{(12)}_{x|y+n}}{\ddot{a}_{xy}}$, the extra annual premium for the continuous instalments. This expression involves the assumption that both (x) and (y) are alive on the valuation date. If only (x) is alive, the reserve is simply $\ddot{a}^{(12)}_{\overline{n}|}\cdot{}_tV(\bar{A}_x)$.

7. Compound reversionary annuities

Although rarely encountered in practice, compound reversionary annuities are of theoretical interest since they represent the annuity counterpart of the compound contingent insurances discussed in the preceding chapter. Under such an annuity, payments are made only in the event that the antecedent deaths have occurred in a prescribed order. Many functions of this type may be defined, and, since little classification is possible, each particular case should be approached from first principles.

The symbol $\bar{a}^{1}_{yz|x}$ denotes the present value of a continuous annuity of 1 per annum payable to (x) commencing upon the death of (y) provided that (y) dies before a third life (z). This benefit may

be regarded as an insurance contingent on (y) being the first of the three lives to die, with the amount of insurance at any time being equal to the present value of a continuous annuity at (x)'s attained age. We thus have

$$\bar{a}^{1}_{yz|x} = \int_0^\infty v^t {}_tp_{xyz}\mu_{y+t}\bar{a}_{x+t}\, dt. \tag{13.13a}$$

An equivalent definite integral may be written by considering the function from another point of view. Annuity payments will be made at time t provided that (x) is then alive and that (y) has previously died before (z). We may thus write

$$\bar{a}^{1}_{yz|x} = \int_0^\infty v^t {}_tp_x \cdot {}_tq^{1}_{yz}\, dt. \tag{13.13b}$$

When $\bar{a}^{1}_{yz|x}$ is known, the value of $\bar{a}^{1}_{yz|x}$ may be obtained from the relation

$$\bar{a}^{1}_{yz|x} + \bar{a}^{1}_{yz|x} = \bar{a}_{yz|x}\,. \tag{13.14}$$

The symbol $\bar{a}^{2}_{yz|x}$ denotes the present value of an annuity to (x) commencing upon the death of (y) provided that (y) dies after (z). This value is given by

$$\begin{aligned}\bar{a}^{2}_{yz|x} &= \int_0^\infty v^t {}_tp_{xy}(1 - {}_tp_z)\mu_{y+t}\bar{a}_{x+t}\, dt \\ &= \bar{a}_{y|x} - \bar{a}^{1}_{yz|x}\,. \end{aligned} \tag{13.15}$$

The relation

$$\bar{a}^{1}_{yz|x} + \bar{a}^{2}_{yz|x} = \bar{a}_{y|x}$$

is clearly correct.

Another way of expressing $\bar{a}^{2}_{yz|x}$ is by regarding the benefit as an insurance contingent upon (z) dying first with the amount of insurance at any time equal to the current present value of a reversionary annuity to (x) after (y). This point of view leads to the following result:

$$\bar{a}^{2}_{yz|x} = \int_0^\infty v^t {}_tp_{xyz}\mu_{z+t}\bar{a}_{y+t|z+t}\, dt = \bar{a}^{1}_{yz|x} - \bar{a}_{z|xy}\,. \tag{13.16}$$

The student should give a verbal interpretation of this result and show that it is equivalent to the one obtained above.

In the case of more complex functions, a notation similar to that used with compound contingent insurances can be adopted. A numeral is written above the life upon whose failure the benefit becomes payable, and numerals written below indicate the relative order of death of the other lives. For example, $\bar{a}^{2}_{\underset{1}{xyz|w}}$ represents an annuity to (w) commencing upon the death of (x) provided that (x) dies before (z) and after (y):

$$\bar{a}^{2}_{\underset{1}{xyz|w}} = \int_0^\infty v^t {}_tp_{wxz}(1 - {}_tp_y)\mu_{x+t}\bar{a}_{w+t}\,dt = \bar{a}^{1}_{\underset{}{xz|w}} - \bar{a}^{1}_{\underset{}{xyz|w}} \quad (13.17)$$

In the two examples that follow, the student should describe the benefit and verify the expressions given:

$$\bar{a}^{2:3}_{\underset{1}{x\ yz|w}} = \int_0^\infty v^t {}_tp_{wxyz}\mu_{y+t}\bar{a}_{x+t|w+t}\,dt = \bar{a}^{1}_{\underset{}{xyz|w}} - \bar{a}^{1}_{\underset{}{yz|wx}} \quad (13.18)$$

$$\bar{a}^{4}_{\underset{21}{wxyz|v}} = \int_0^\infty v^t {}_tp_{vwxy}(1 - {}_tp_z)\mu_{y+t}\bar{a}^{2}_{\overline{w+t}:x+t|v+t}\,dt \quad (13.19)$$

It is not necessary to give special attention to compound reversionary annuities in non-continuous form since, just as with simple reversionary annuities, the values of the discrete annuities are approximately the same as the values of the corresponding continuous annuities.

8. Summary of notation

A vertical bar separating the elements of a suffixed subscript indicates a reversionary function, the status after the bar being understood to follow the status before the bar. Thus $a^{(m)}_{y|x}$ is an annuity on the life of (x) after the death of (y) with the first annuity payment of $1/m$ due at the end of the $1/m$-th part of a year in which (y) dies, the payment periods being measured from the date of issue of the contract. When the payment periods are measured from the date of death, the distinguishing symbol $\hat{a}^{(m)}$ is used.

References

1–5. Similar material is covered by Spurgeon (1932), Chapter 18, and by Hooker and Longley-Cook (1957), Chapter 25.

6. Policies with continuous instalments are discussed by Bergstresser (1941).

7. Compound reversionary annuities are treated by Spurgeon (1932), Chapter 19.

EXERCISES

1, 2. The basic function

1. Explain by general reasoning and show algebraically:

$$a_{y|x} = \sum_{t=1}^{\infty} v^t {}_tp_x \cdot {}_{t-1|}q_y \ddot{a}_{x+t} .$$

2. Express in terms of single- and joint-life annuities:
   (a) $a_{z|\overline{xy}}$
   (b) $a_{\overline{yz}|wx}$.

3. The last-survivor annuity $a_{\overline{xyz}}$ is payable in the following way. While all three lives are surviving, (x) and (y) each receive $\frac{1}{4}$ of the annual payment and (z) receives $\frac{1}{2}$. Upon the death of (x), his share passes to (y), if surviving, otherwise to (z). Upon the death of either (y) or (z), their shares are divided equally among the survivors, the last survivor receiving the total annuity payment of 1. Find expressions for the present values of the individual shares of (x), (y), and (z) in terms of single- and joint-life annuities.

3. Modifications of the basic function

4. Give two independent integral expressions for $\bar{a}_{\overline{xy}|z}$. Demonstrate their equivalence by reducing both to the same expression in terms of simple continuous annuities.

5. Show that

(a) $$\frac{\partial}{\partial x} \bar{a}_{y|x} = \mu_x \bar{a}_{y|x} - \bar{A}^{\,2}_{xy}$$

(b) $$\bar{A}^{\,2}_{xy} = \bar{A}^{\,1}_{xy} - \delta \bar{a}_{y|x} .$$

6. Show that

$$\hat{a}^{(m)}_{y|x} \doteq \bar{a}_{y|x} + \frac{1}{2m} \bar{A}^{\,1}_{xy} .$$

4. Functions involving terms-certain

7. Express in terms of commutation functions and simple annuity values:
   (a) the present value of a temporary annuity of 1 payable to (x) with the first payment at the end of n years from the end of the year of death of (y), but with no payments to be made after m years from the present time $(m > n)$;
   (b) the present value of a temporary annuity of 1 payable to (x) during the lifetime of (y) and for n years after the death of (y), but with no payments to be made after m years from the present time $(m > n)$.

8. An annuity of 1 becomes payable at the end of the year of death of (x) and continues until n years after the death of (y). Express the present value in terms of net single premium symbols on each of the following assumptions:

(a) the annuity is entered upon only if (y) is alive at the death of (x);

(b) the annuity is payable whether or not (y) is alive at the death of (x).

9. Find an expression in a form free from integrals or summations for the value of an annuity of \$100 per annum payable to (y), the first payment to be made at the end of the t-th year succeeding the year in which (x) dies, provided (x) dies within n years, the annuity to be void if (x) lives beyond n years.

10. For each of the following, describe the benefit and give a simple formula (using standard approximations if necessary):

(a) $a_{y|\overline{x:n}|}$, (b) $a^{(4)}_{\overline{x:n}|y}$.

5. Annual premiums and reserves

11. Find as closely as possible the net annual premium for a reversionary annuity of 1 per annum to a life aged 40 after the death of another life aged 43, the first annuity payment to be a full payment of 1 and to be made on the anniversary of the date of issue, given the following data from a Gompertz table:

x	a_x	μ_x	x	a_x	μ_x
40	15.136	.00990	54	11.331	.01950
43	14.413	.01106	55	11.024	.02077
51	12.230	.01631	56	10.714	.02216
52	11.935	.01727	62	8.808	.03381
53	11.635	.01833	63	8.488	.03645

(Ans. .347)

12. Show that ${}_tV(a_{y|x})$ is negative if

$$\frac{\ddot{a}_{x+t}}{\ddot{a}_x} < \frac{\ddot{a}_{x+t:y+t}}{\ddot{a}_{xy}}.$$

13. Express in annuity symbols the reserve t years after issue for a single premium reversionary annuity payable to a life (z) after the death of the survivor of two lives (x) and (y).

6. Continuous instalments

14. An m-year endowment policy with continuous instalments provides that if the insured, aged x at issue, dies before the maturity of the policy as an endowment, a monthly income will be paid for n years certain and for as long thereafter as the beneficiary, aged y at issue, survives. At maturity as an endowment the income is payable for n years certain and as long thereafter as either the insured or beneficiary survives.

Show that the net single premium for the policy consists of the following four parts:

(a) for the annuity certain: $\ddot{a}^{(12)}_{\overline{n}|}\,\bar{A}_{x:\overline{m}|}$

(b) for the continuous instalments to the beneficiary in the event of the insured's death prior to maturity:

$$\frac{D_{y+n}}{D_y}\left(\hat{\ddot{a}}^{(12)}_{x|y+n} - \frac{D_{x+m:y+n+m}}{D_{x:y+n}}\,\hat{\ddot{a}}^{(12)}_{x+m|y+n+m}\right)$$

(c) for the continuous instalments to the insured after maturity:

$${}_{m+n|}\bar{a}^{(12)}_{x}$$

(d) for the continuous instalments to the beneficiary in the event of the insured's death after maturity:

$$({}_mp_x - {}_{m+n}p_x)\,\frac{D_{y+m+n}}{D_y}\,\bar{a}^{(12)}_{y+m+n} + \frac{D_{x+m+n:y+m+n}}{D_{xy}}\,\ddot{a}^{(12)}_{x+m+n|y+m+n}$$

15. In Exercise 14, if the $\hat{\ddot{a}}^{(12)}$ functions are replaced by $\ddot{a}^{(12)}$, show that the sum of parts (b) and (d) of the net single premium reduces to

$${}_{n|}\ddot{a}^{(12)}_{y} - {}_{m+n|}\ddot{a}^{(12)}_{xy} - \frac{D_{y+n}}{D_y}\,\ddot{a}^{(12)}_{x:y+n:\overline{m}|}\,.$$

16. In the policy described in Exercise 14, the annual premiums are arranged so that if the beneficiary dies during the premium-paying period, the premium will reduce to that for a policy on the insured's life only. Using the approximation indicated in Exercise 15, write the formula for the net annual premium.

7. Compound reversionary annuities

17. When Gompertz's law holds, show that

$$\bar{a}^{1}_{yz|x} = {}_{\infty}q^{1}_{yz}\cdot\bar{a}_{yz|x}\,.$$

18. Give two different expressions in the form of integrals for the value of a continuous annuity of 1 per annum, payable so long as at least one of two lives aged x and y shall survive after the death of z, provided z predeceases a fourth life w, and show that your expressions are equal.

19. Show that $\bar{a}^{1}_{xy|z} = \int_0^\infty v^t\,{}_tp_z\bar{A}^{1}_{xy:z+t}\,dt$.

Miscellaneous applications

20. Develop a formula in terms of annuity and insurance symbols for the gross single premium to provide the following benefits with a loading of 6½% of the gross premium: A last-survivor annuity on the lives of a husband and wife, deferred n years and reducing by one-third on the first death. If the husband's death occurs prior to that of the wife and during the de-

ferred period, the annuity on the reduced basis is to commence immediately to the wife. If the husband's death occurs after that of the wife and during the deferred period, the single premium is to be refunded.

21. A man aged 35 has a wife aged 30 and two children aged 5 and 10, respectively. Under an insurance program, upon the man's death while either of his two children is under age 18 a monthly annuity will become payable subject to the following conditions:

(i) \$30 monthly payable to his wife as long as she survives and either child survives and has not attained age 18,

(ii) \$20 monthly payable to each child as long as the child survives and has not attained age 18, payable whether or not the man's wife is alive at the time the payments come due.

Assuming that the monthly functions may be replaced by continuous functions without appreciable loss of accuracy, express the value of the insurance program in terms of continuous life annuities.

22. An insured holds a \$5000 policy, whole life fully paid up at age 65, issued at age x five years ago. An option in the policy provides for settlement in equal monthly instalments for a period of twenty years, the first payment to be made at the date of death, and such that the present value of the instalments at the date of death is \$5000. The insured now desires to provide for an additional benefit, whereby the monthly instalments will continue during the lifetime of the beneficiary now aged y. Find an expression for the additional net annual premium, assuming such premiums are not to be payable after the death of the beneficiary. Give an expression for the reserve on the additional benefit after five years, assuming both the insured and the beneficiary are then alive.